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A new approach for the Modified Local Green's Function Method applied to Poisson equation

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Abstract. Originally proposed at the end of the 80 's, the Modified Local Green's Function Method (MLGFM) is an integral method that was described as a hybrid of the Finite Element Method (FEM) and the Boundary Element Method (BEM). The method was proposed to apply the BEM methodology to problems with no knowledge of the fundamental solution. Essentially, the MLGFM creates discrete projections of the Green's function solving an auxiliary domain problem, and this problem can be solved, for example, by the FEM formulation. Despite the good convergence of the secondary variable in the boundary, the method has a major disadvantage over FEM, the obtainment of the Green's function projections implies in to solve the system of equations in the domain, resulting in a great computational effort. However, this paper aims to show a new approach to the MLGFM where it is not necessary to obtain the Green's projections and the final equations system has the same number of degrees of freedom as FEM and still presents high convergence for the secondary variable in the boundary. The new formulation can be obtained directly by the weighted residual sentence and variables using the same approximation of the original MLGFM. The processing time of the two approaches are compared and the method is applied to Poisson Equation.

Keywords: Green Functions, Modified Local Green's Function Method, Finite Element Method, Boundary Element Method

1 Introduction

Originally proposed in Barcellos and Silva [1], the Modified Local Green's Function Method (MLGFM) is an integral method described as a hybrid of the Finite Element Method (FEM) and the Boundary Element Method (BEM). The method was proposed to apply the BEM methodology to problems with no knowledge of the fundamental solution. Essentially, the MLGFM creates discrete projections of the Green's function solving an auxiliary domain problem, its problems can be solved, for example, by the FEM.

The MLGFM has been applied to solve several continuum mechanics problems over the years, for example, membranes by Barcellos and Silva [1], singular Poisson problems by Barcellos and Barbieri [2], *h*- and *p*-convergence of Helmholtz equation by Filippin [3], Mindlin's plates by Barbieri and Barcellos [4], non-homogeneous potentials problems by Barbieri and Barcellos [5], laminated plates by Machado and Barcellos [6], Machado et al. [7], and Machado et al. [8], for 3D elasticity by Barcellos et al. [9], and Anisotropic Heat Conduction by Muñoz-Rojas and Vaz Jr. [10]. According to Barbieri and Muñoz [11], the MLGFM demonstrated great accuracy and high convergence rates, especially in terms of fluxes on the boundary.

Despite the good convergence of the flux variables in the boundary, the method has a major disadvantage over FEM, the obtainment of the Green's function projections implies solving many systems of equations in the domain, resulting in a great computational effort. This fact will become clearer in MLGFM equations in the next sections. However, this paper aims to show a new approach where it is not necessary to obtain Green's projections and the final equations system has the same number of degrees of freedom as FEM. The new approach of the method still presents high convergence for the flux variables in the boundary.

2 The MLGFM original formulation

Assuming there is a differential operator L and two associated operators D and N related to Dirichlet and Neumann boundary conditions, respectively. The method consists of finding the Green tensor projections solving an auxiliary problem in domain related to adjunct operators L^* and N^*. To solve this auxiliary problem is necessary to define an additional operator N' (for more details see Barbieri et al. [12]) and define a new variable:

$$f(p) = (N + N')\, u(p), \tag{1}$$

where $u(p)$ is the potential in boundary.

The additional operator allows writing the system of integral equations as:

$$u(Q) = \int_\Omega \mathbf{G}^T(P,Q)\, b(P)\, d\Omega_P + \int_\Gamma \mathbf{G}^T(p,Q)\, f(p)\, d\Gamma_p; \quad Q, P \in \Omega \text{ and } p \in \Gamma, \tag{2}$$

where $\mathbf{G}(\cdot,\cdot)$ is the Green's tensor, $b(P)$ is the domain source, $f(p)$ is the flux in boundary, $u(Q)$ is the potential in domain and P and Q are a source point and a field point in domain, respectively.

To extend this integral equation to boundary, the trace operator is applied:

$$u(q) = \int_\Omega \mathbf{G}^T(P,q)\, b(P)\, d\Omega_P + \int_\Gamma \mathbf{G}^T(p,q)\, f(p)\, d\Gamma_p; \quad P \in \Omega \text{ and } p, q \in \Gamma, \tag{3}$$

were $u(q)$ is the potential in boundary and p and q are a source point and a field point in boundary, respectively.

Approximating domain and boundary variables using domain $\mathbf{\Psi}$ and boundary $\mathbf{\Phi}$ basis functions, in the form:

$$\begin{aligned} u(Q) &= \mathbf{\Psi}(Q)\, \mathbf{u}_\mathrm{D} \qquad & u(q) &= \mathbf{\Phi}(q)\, \mathbf{u}_\mathrm{B} \\ b(P) &= \mathbf{\Psi}(P)\, \mathbf{b} \qquad & f(q) &= \mathbf{\Phi}(p)\, \mathbf{f} \end{aligned} \quad . \tag{4}$$

where $\mathbf{u}_\mathrm{D}$ and $\mathbf{u}_\mathrm{B}$ are vectors containing the nodal values of the function u in domain and in boundary, respectively; $\mathbf{b}$ is a vector containing the nodal values of the domain source $b(p)$; and $\mathbf{f}$ is a vector containing the nodal values of the normal flux in the boundary.

The basis functions are the FEM and BEM traditional functions. Here these functions are the bilinear FEM shape for domain and linear functions for boundary. Note that the boundary functions need to be a trace of the domain functions, or:

$$\mathbf{\Phi}(q) = \lim_{Q \to q} \mathbf{\Psi}(Q); \qquad q \in \Gamma \text{ and } Q \in \Omega. \tag{5}$$

The equations (2) and (3) can be rewritten as:

$$\mathbf{A}\, \mathbf{u}_\mathrm{D} = \mathbf{B}\, \mathbf{f} + \mathbf{C}\, \mathbf{b}, \tag{6}$$

$$\mathbf{D}\, \mathbf{u}_\mathrm{B} = \mathbf{E}\, \mathbf{f} + \mathbf{F}\, \mathbf{b}, \tag{7}$$

where:

$$\mathbf{A} = \int_\Omega \mathbf{\Psi}^T(Q)\, \mathbf{\Psi}(Q)\, d\Omega_Q; \tag{8}$$

$$\mathbf{B} = \int_{\Omega} \mathbf{\Psi}^T(Q)\, \mathbf{G}_{\mathrm{b}}(Q)\, d\Omega_Q; \tag{9}$$

$$\mathbf{C} = \int_{\Omega} \mathbf{\Psi}^T(Q)\, \mathbf{G}_{\mathrm{d}}(Q)\, d\Omega_Q; \tag{10}$$

$$\mathbf{D} = \int_{\Gamma} \mathbf{\Phi}^T(q)\, \mathbf{\Phi}(q)\, d\Gamma_q \tag{11}$$

$$\mathbf{E} = \int_{\Gamma} \mathbf{\Phi}^T(q)\, \mathbf{G}_{\mathrm{b}}(q)\, d\Gamma_q \tag{12}$$

$$\mathbf{F} = \int_{\Gamma} \mathbf{\Phi}^T(q)\, \mathbf{G}_{\mathrm{d}}(q)\, d\Gamma_q \tag{13}$$

where $\mathbf{G}_{\mathrm{b}}(Q)$, $\mathbf{G}_{\mathrm{d}}(Q)$, $\mathbf{G}_{\mathrm{b}}(q)$ and $\mathbf{G}_{\mathrm{d}}(q)$ are the Green's function projections over the boundary Γ and the domain Ω, evaluated on the points Q and q. The Green's projections can be written as:

$$\mathbf{G}_{\mathrm{b}}(Q) = \int_{\Gamma} \mathbf{G}^T(p, Q)\, \mathbf{\Phi}(q)\, d\Gamma_p, \tag{14}$$

$$\mathbf{G}_{\mathrm{d}}(Q) = \int_{\Omega} \mathbf{G}^T(P, Q)\, \mathbf{\Psi}(P)\, d\Omega_p, \tag{15}$$

$$\mathbf{G}_{\mathrm{b}}(q) = \int_{\Gamma} \mathbf{G}^T(p, q)\, \mathbf{\Phi}(q)\, d\Gamma_p, \tag{16}$$

$$\mathbf{G}_{\mathrm{d}}(q) = \int_{\Omega} \mathbf{G}^T(P, q)\, \mathbf{\Psi}(P)\, d\Omega_p. \tag{17}$$

The Green's tensor projection can be also projected in the space formed by the domain and the boundary shape functions:

$$\begin{aligned} \mathbf{G}_{\mathrm{d}}(Q) &= \mathbf{\Psi}(Q)\, \mathbf{G}^{\mathrm{DQ}} \qquad & \mathbf{G}_{\mathrm{d}}(q) &= \mathbf{\Phi}(q)\, \mathbf{G}^{\mathrm{Dq}} \\ \mathbf{G}_{\mathrm{b}}(Q) &= \mathbf{\Psi}(Q)\, \mathbf{G}^{\mathrm{BQ}} \qquad & \mathbf{G}_{\mathrm{b}}(q) &= \mathbf{\Phi}(q)\, \mathbf{G}^{\mathrm{Bq}} \end{aligned} . \tag{18}$$

where $\mathbf{G}^{\mathrm{DQ}}$, $\mathbf{G}^{\mathrm{Dq}}$, $\mathbf{G}^{\mathrm{BQ}}$ and $\mathbf{G}^{\mathrm{Bq}}$ are the nodal coefficients of the Green Tensors project in the space formed by domain and boundary shape functions. Its values can be determined by the minimization of an appropriate functional (see Barbieri et al. [12]), resulting in the expression:

$$\left[\mathbf{K}_{\mathrm{FEM}} + \mathbf{K}'\right] \left[\mathbf{G}^{\mathrm{DQ}} \,|\, \mathbf{G}^{\mathrm{BQ}}\right] = \left[\mathbf{A} \,|\, \mathbf{D}\right], \tag{19}$$

where $\mathbf{K}_{\mathrm{FEM}}$ is the FEM stiffness matrix, $\mathbf{K}'$ is a diagonal matrix related to additional operator N', $\mathbf{A}$ and $\mathbf{D}$ are the matrix defined in (8) e (11). The values of $\mathbf{G}^{\mathrm{Dq}}$ and $\mathbf{G}^{\mathrm{Bq}}$ can be determined by the application of the trace operator in $\mathbf{G}^{\mathrm{DQ}}$ and $\mathbf{G}^{\mathrm{BQ}}$.

Once the Green's functions are obtained, all the matrices necessary for the solution of the eq. (7) are obtained and the boundary values are obtained. The domain values can be obtained directly with the following expression:

$$\mathbf{u}_{\mathrm{D}} = \mathbf{G}^{\mathrm{BQ}}\mathbf{f} + \mathbf{G}^{\mathrm{DQ}}\mathbf{b}. \tag{20}$$

3 The MLGFM new formulation

Applying the Green's theorem in a generically variational boundary-value problem, its possible to find a expression like:

$$B\left(u,w\right)-\int_{\Gamma}N\,u\,w\,d\Gamma=\int_{\Omega}b\,w\,d\Omega. \tag{21}$$

where $B\left(\cdot,\cdot\right)$ is a bilinear form associated to the differential operator L and w are the weighting functions.

The expression in eq. (21) is simply the Galerkin weak form. Now defining the boundary flux variable as:

$$f(p)=Nu(p), \tag{22}$$

employing the approximations for the variables define in eq. (18) and using the domain basis functions as weighting functions w, the expression in eq. (21) becomes in:

$$B\left(\mathbf{\Psi}(P),\mathbf{\Psi}(P)\right)\,\mathbf{u}_{\mathrm{D}}=\int_{\Gamma}\mathbf{\Psi}^{T}(p)\,\mathbf{\Phi}(p)\,d\Gamma\,\mathbf{f}+\int_{\Omega}\mathbf{\Psi}(P)^{T}\,\mathbf{\Psi}(P)\,d\Omega\,\mathbf{b}, \tag{23}$$

where the bilinear form $B\left(\cdot,\cdot\right)$ is the same related to the FEM stiffness matrix. Recognizing the matrices $\mathbf{A}$ and the $\mathbf{D}$ (the functions $\mathbf{\Phi}$ are the trace of $\mathbf{\Psi}$), the expression is reduced to:

$$\mathbf{K}_{\mathrm{FEM}}\,\mathbf{u}_{\mathrm{D}}=\mathbf{D}\,\mathbf{f}+\mathbf{A}\,\mathbf{b}, \tag{24}$$

and this system can be solved similarly to the Boundary Element Method.

Disregarding the effects of the additional operator N', its possible to write:

$$\mathbf{K}_{\mathrm{FEM}}\,\mathbf{G}^{\mathrm{DQ}}=\mathbf{A}\quad\text{or}\quad\mathbf{G}^{\mathrm{DQ}}=\mathbf{K}_{\mathrm{FEM}}^{-1}\,\mathbf{A}; \tag{25a}$$

$$\mathbf{K}_{\mathrm{FEM}}\,\mathbf{G}^{\mathrm{BQ}}=\mathbf{D}\quad\text{or}\quad\mathbf{G}^{\mathrm{BQ}}=\mathbf{K}_{\mathrm{FEM}}^{-1}\,\mathbf{D}; \tag{25b}$$

In other words, the Green's functions are implicitly presented in the new formulation final system.

This new approach to the MLGFM avoids the need to find the $\mathbf{G}^{\mathrm{BQ}}$ and $\mathbf{G}^{\mathrm{DQ}}$ tensors. Note that the eq. (19) is composed of several systems of equations since $\mathbf{A}$ and $\mathbf{D}$, in right-hand side of this equation, are matrices, not vectors. The new approach needs to solve only one system of equations and maintains the same properties of the original approach.

4 Numerical model

This example is a square membrane with edge L, subject to a domain source $b(x,y)$. The governing equation of this problem is:

$$\nabla^{2}u\left(x,y\right)=-\frac{x}{L}\,\cos\frac{\pi\,y}{L}; \tag{26}$$

with the follow boundary conditions:

$$u\left(0,y\right)=u\left(L,y\right)=0\quad0\leq y\leq L; \tag{27a}$$

$$\left.\frac{\partial u}{\partial\mathbf{n}}\right|_{y=0}=\left.\frac{\partial u}{\partial\mathbf{n}}\right|_{y=L}=0\qquad0\leq x\leq L. \tag{27b}$$

The analytical solution for potential $u\,(x, y)$ is given by Loeffler et al. [13]:

$$u\,(x, y) = \left[\frac{L}{\pi^2\,\sinh\pi}\,\sinh\frac{\pi\,x}{L} - \frac{x}{\pi^2}\right]\,\cos\frac{\pi\,y}{L}. \tag{28}$$

The error measure for potential in domain is taken as an approximation of L_2-norm error normalized by $|u|_{\max}$, defined in Khan et al. [14] in the form:

$$\epsilon_{er} = \frac{1}{|u|_{\max}}\sqrt{\frac{1}{M}\sum_{i=1}^{M}\left(u_i^{(e)} - u_i^{(n)}\right)^2}, \tag{29}$$

where M is the number of nodes, $|u|_{\max}$ is the maximum value on the M sample nodes, $u_i^{(e)}$ is the exact value on node i and $u_i^{(n)}$ is the approximate value on node i.

Firstly to show the difference of processing times between the old and the new approach, the processing time for models with increasing number of elements in the mesh are presented in Fig. 1. The computational routines are implemented using the Python© language.

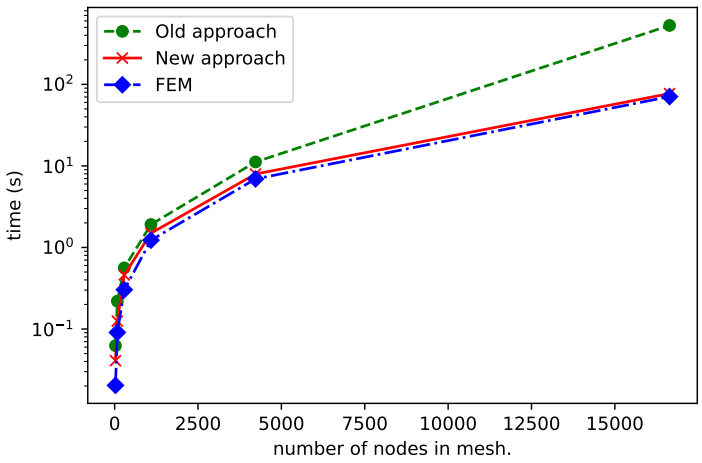

Figure 1. Comparison of processing time of two approaches of the MLGFM.

Figure 1 shows that as more refined the mesh, greater the savings in processing time for the new approach. The new approach equation system has the same numbers of degrees of freedom as FEM, which makes the new approach more competitive compared to the original formulation.

The comparison between MFGLM and FEM is presented in Fig. 2. As can be observed the errors of MLGFM are smaller than FEM ones and the convergence rate is almost the same.

The flux in x-direction on the face with $x = 1$ is obtained directly in MLGFM by solving the system in eq. (24) and in FEM a Least Square procedure is used to recover the flux. The comparison between the MLGFM and FEM for a coarse mesh (5x5) is presented in Fig. 3. As can be observed, even for a coarse mesh the nodal values of flux in MLGFM are almost exact. Mendonça [15] demonstrated that if the nodal values of Green's functions obtained in MLGFM are exact the flux results are also exact.

5 Final remarks

A new approach for the MLGFM is presented in this paper. The new approach reduces significantly the processing time of the MLGFM original formulation. This new approach avoids obtaining the Green's projections, reducing the original processing time.

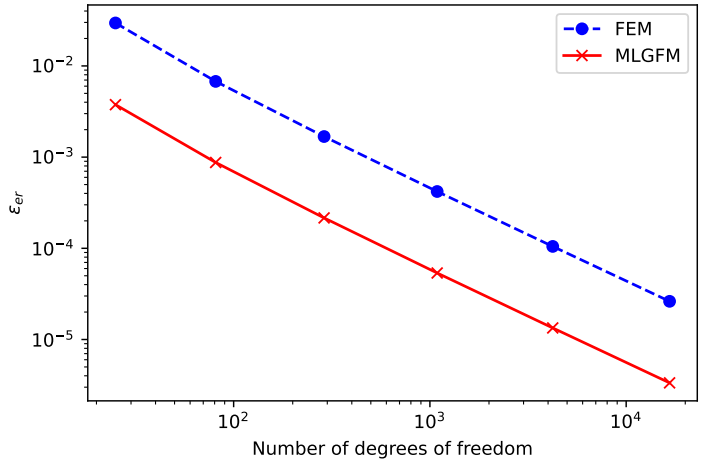

Figure 2. Comparison between MLGFM and FEM on L_2 error norm for domain source example.

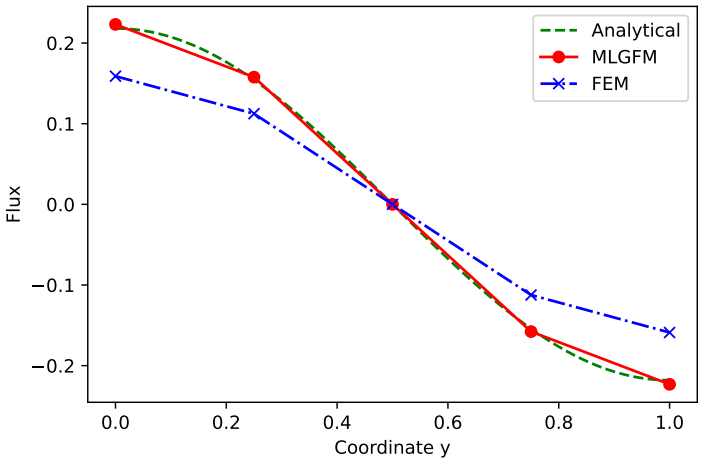

Figure 3. Comparison between MLGFM and FEM flux results in x-direction for domain source example.

The new approach processing time is similar to the FEM, so the MLGFM is more competitive when compared to the FEM. The new formulation maintains the main feature of the MLGFM, presenting good convergence of the flux variables in the boundary of the studied domain. The next step of this research is to bring the enrichment technique, based on GFEM, to the new approach of the MLGFM.

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