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A reducer order model approach for fuzzy field seepage analysis

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Abstract. This contribution proposes a strategy for performing seepage analysis where uncertainty associated with permeability is characterized by means of fuzzy fields. In order to decrease numerical costs associated with uncertainty propagation, full system analysis is replaced by a reduced order model. This reduced order model projects the equilibrium equations to a small-dimensional space, which is constructed using a single analysis of the system plus a sensitivity analysis. The associated basis is enriched to ensure the quality of the approximate response. A simple numerical example shows that with the presented strategy, it is possible to accurately estimate the fuzzy seepage flow with reduced numerical efforts.

Keywords: Fuzzy field, seepage analysis, reduced order model.

1 Introduction

Seepage analysis is of utmost importance in several practical engineering problems such as dam design. In this context, seepage can be quantified by resorting to, e.g., finite elements. However, the parameters that govern the seepage phenomenon, such as permeability, can be seldom quantified precisely (see Baroni et al. [1]). Several difficulties are encountered in practice when characterizing permeability, such as imprecision and scarcity in field measurements, anisotropy in vertical and horizontal directions, and spatial variation. Under such a scenario, fuzzy fields appear as a viable tool for treating problems that exhibit imprecision with a spatial component. Nonetheless, dealing with fuzzy fields imposes a major challenge, as it becomes necessary to propagate uncertainty considering spatial dependencies, which can be quite challenging when large-scale numerical models are involved. This difficulty stems directly from (1) the by-definition orthogonality between any two intervals in a set of intervals; (2) the computational burden associated with the multiple calls to the model.

This contribution proposes an approach for seepage analysis where uncertainty associated with permeability is characterized by means of fuzzy fields. In this context, fuzzy fields are defined as a natural extension of the Inverse Distance Weighting framework that is commonly applied in an interval field context (see Faes and Moens [2]). Concerning the propagation of the fuzzy field, the traditional alpha-level optimization strategy is adopted for calculating the membership function associated with seepage flow in a discrete manner (see Moeller et al. [3]). Within this optimization process and to decrease numerical costs, full finite element analyses are replaced by a reduced order model that projects the system's equations to a small-dimensional space. The basis associated with the reduced order model is constructed by means of a single analysis of the system plus a sensitivity analysis. This reduced basis is enriched adaptively as the alpha-level optimization strategy progresses to protect the quality of the approximations provided by the reduced order model. A numerical example illustrates that the proposed strategy allows for characterizing the fuzzy seepage flow with improved numerical efficiency.

2 Formulation of the Problem

2.1 Finite element formulation

The partial differential equation that governs the 2D steady-state confined seepage problem under uncertainty is:

$$\frac{\partial}{\partial x}\left(k_H\left(\boldsymbol{\theta}\right)\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_V\left(\boldsymbol{\theta}\right)\frac{\partial h}{\partial y}\right)=0 \tag{1}$$

where $\boldsymbol{\theta}$ represents a vector that collects the uncertain parameters; k_H and k_V correspond to the horizontal and vertical permeabilities, respectively; and h is the hydraulic head.

Equation 1 is solved by means of the finite element (FE) method,

$$\boldsymbol{K}\left(\boldsymbol{\theta}\right)\boldsymbol{H}\left(\boldsymbol{\theta}\right)=\boldsymbol{Q}\left(\boldsymbol{\theta}\right) \tag{2}$$

where $\boldsymbol{K}\left(\boldsymbol{\theta}\right)$ is the matrix associated with the soil permeabilities; $\boldsymbol{Q}\left(\boldsymbol{\theta}\right)$ is the vector representing nodal flow; and $\boldsymbol{H}\left(\boldsymbol{\theta}\right)$ is the vector that describes the system's response, that is, hydraulic head. As noted from eq. (2), the uncertainty affecting the system's matrix $\boldsymbol{K}\left(\boldsymbol{\theta}\right)$ and the flow $\boldsymbol{Q}\left(\boldsymbol{\theta}\right)$ propagates to the hydraulic head $\boldsymbol{H}\left(\boldsymbol{\theta}\right)$.

2.2 Fuzzy Fields

A possible way of quantifying the uncertainty associated with the system's response corresponds to applying techniques of fuzzy analysis (see Beer et al. [4]). Thus, each uncertain parameter $\theta_i, i=1,\ldots,n_\theta$, where n_θ indicates the number of uncertain parameters; can be characterized as a fuzzy variable $\tilde{\theta}_i$. In this context, a fuzzy variable can be interpreted as a collection of intervals for different membership levels, where these intervals are indexed by a membership function $\mu_{\tilde{\theta}_i}(\theta_i)\in[0,1]$. Consequently, as the input parameters are characterized as fuzzy variables, the response will depend on the membership level under analysis, which implies that the system's response is a fuzzy variable as well.

One approach to determine the membership function of the response is to use the previous interpretation of a fuzzy variable. For this purpose, the membership functions $\mu_{\tilde{\theta}_i}(\theta_i)$ associated to each fuzzy variable $\tilde{\theta}_i$ are analyzed for discrete membership values α_j, with $j=1,\ldots,n_c$, where n_c indicates the number of discrete levels considered. This implies that for each of these membership levels α_j, there will be an interval associated with each variable θ_i:

$$\underline{\theta}_{i,\alpha_j}=\left\{\theta_i\in\Theta_i:\mu_{\tilde{\theta}_i}\left(\theta_i\right)\geq\alpha_j\right\}, i=1,\ldots,n_\theta,\alpha_j\in(0,1] \tag{3}$$

where $\underline{\theta}_{i,\alpha_j}$ represents the possible set of values that θ_i can assume for an α_j-cut of the membership function. Note that this cut corresponds to an interval whose lower and upper limits are θ^L_{i,α_j} and θ^R_{i,α_j}, respectively.

For a given membership level α_j, the response of interest r will be contained in an interval $\underline{r}_{\alpha_j}$ with lower $r^L_{\alpha_j}$ and upper $r^R_{\alpha_j}$ bounds. That is,

$$r^L_{\alpha_j}=\min_{\boldsymbol{\theta}}(r(\boldsymbol{\theta})),\theta_i\in\underline{\theta}_{i,\alpha_j}, i=1,\ldots,n_\theta \tag{4}$$

$$r^R_{\alpha_j}=\max_{\boldsymbol{\theta}}(r(\boldsymbol{\theta})),\theta_i\in\underline{\theta}_{i,\alpha_j}, i=1,\ldots,n_\theta \tag{5}$$

In this contribution, the response of interest r corresponds to the seepage total flow, where the lower $r^L_{\alpha_j}$ and upper $r^R_{\alpha_j}$ bounds, from eq. (4) and eq. (5), respectively, are determined by optimization.

The characterization of input parameters by means of fuzzy variables, such as those described recently, assumes that parameters under study are only affected by uncertainty. Nevertheless, for some variables, such as soil properties, uncertainty is also dependent on spatial coordinates. Therefore, in the presence of spatial dependencies, the fuzzy variable concept can be extended as a fuzzy field.

Consider the case presented in Fig. 1, where a single input variable θ exhibits spatial dependence. In that case, for simplicity, there is information (i.e., physical measures) of the input parameter at two specific locations on the domain Ω (x_1 and x_2 in Fig. 1). These positions correspond to the control points . In each control point, it is possible to characterize the parameter as a fuzzy variable (blue membership functions in Fig. 1). For a specific membership level α_j, note that it is possible to associate an interval as in eq. (3), but now at each location. Using base functions $\boldsymbol{\Psi}$ (see Faes and Moens [2]), the information in the control points is projected to any position on the domain. As a result of that procedure, the interval field associated with α_j is obtained (red area in Fig. 1). If this process is repeated for different membership levels, it discretely approximates the input parameters as a fuzzy field.

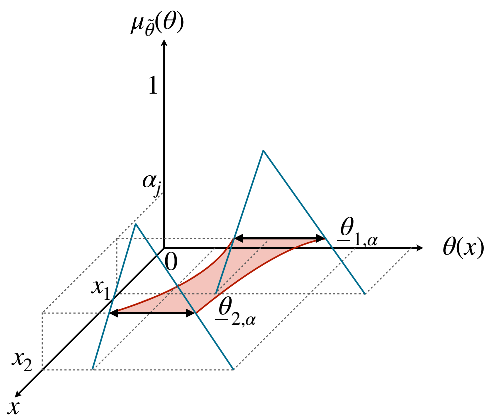


Figure 1. Fuzzy field schematic representation.

Note that: (1) this strategy is appropriate when there is limited data of an input parameter of the model at specified locations on the domain, and (2) fuzzy fields allow propagating the uncertain input information to the finite element mesh. If it is considered as a representative coordinate of each finite element its centroid, then the information of the uncertain parameters propagated to these coordinates $\boldsymbol{\theta}_C$ is given by,

$$\boldsymbol{\theta}_C = \boldsymbol{\Psi}(\boldsymbol{X}_C, \boldsymbol{X}_b)\boldsymbol{\theta}_b \tag{6}$$

where $\boldsymbol{X}_C$ is a matrix whose columns contains the coordinates of each finite element centroid; $\boldsymbol{X}_b$ is a matrix whose columns contains the coordinates of the control point; $\boldsymbol{\Psi}(\boldsymbol{X}_C, \boldsymbol{X}_b)$ is a matrix that contains the base functions information, and $\boldsymbol{\theta}_b$ contains the interval information of the uncertain parameters at control points. From eq. (6), is clear that the uncertainty in the input parameters is reduced to the information contained in the control points. The components of the matrix $\boldsymbol{\Psi}$ are derived from the eq. (7),

$$\psi_j(\boldsymbol{x}, \boldsymbol{X}_b) = \frac{w_j(\boldsymbol{x}, \boldsymbol{x}_{b,j})}{\sum_{j_1=1}^{n_b} w_{j_1}(\boldsymbol{x}, \boldsymbol{x}_{b,j_1})},\ j = 1, \ldots, n_b \tag{7}$$

where ψ_j is the j-th base function with $j = 1, \ldots, n_b$, wherein n_b corresponds to the number of control point considered in Ω; $w_j(\boldsymbol{x}, \boldsymbol{x}_{b,j})$ is the weight function between a specific spacial coordinate $\boldsymbol{x}$ and the node location vector $\boldsymbol{x}_{b,j}$. In this proposal, the weight functions w_j correspond to the inverse distance weighting function (see Faes and Moens [2]).

2.3 Reduced order model

Direct solution of eqs. (4) and (5) can be quite demanding from a numerical viewpoint, as they demand repeated evaluation of the equilibrium equation (see eq. (2)). A possible means to decrease numerical costs consists of applying a reduced order model. Thus, the approximate response in terms of hydraulic head $\boldsymbol{H}^A(\boldsymbol{\theta})$, dependent on the considered fuzzy fields, can be expressed as the linear combination of several known components. These components constitute the reduced basis $\boldsymbol{\Phi}$, which is constructed by means of a single exact analysis of the system plus a sensitivity analysis of the response concerning the uncertain parameters (see Valdebenito et al. [5]). This sensibility analysis demands performing a single system evaluation as well. As part of this sensibility analysis, the partial derivatives are calculated analytically using a direct method (see Haftka and Gürdal [6]) and evaluated at a nominal point $\boldsymbol{\theta}^0$. The nominal point satisfies that $\mu_{\tilde{\theta}_i}(\theta_i^0) = 1, i = 1, \dots, n_\theta$.

The expression to obtain the approximate hydraulic head $\boldsymbol{H}^A(\boldsymbol{\theta})$ is given by:

$$\boldsymbol{H}(\boldsymbol{\theta}) \approx \boldsymbol{H}^{\mathrm{A}}(\boldsymbol{\theta}) = \boldsymbol{\Phi}\boldsymbol{\beta}(\boldsymbol{\theta}) \tag{8}$$

where $\boldsymbol{\beta}(\boldsymbol{\theta})$ is a vector whose components depend on the uncertain parameters. From eq. (2), which admits the characterization of the input parameters as fuzzy fields, and using the reduced basis $\boldsymbol{\Phi}$, the reduced system corresponds to:

$$\boldsymbol{K}_R(\boldsymbol{\theta})\boldsymbol{\beta}(\boldsymbol{\theta}) = \boldsymbol{Q}_R(\boldsymbol{\theta}) \tag{9}$$

where $\boldsymbol{K}_R(\boldsymbol{\theta})$ is the stiffness matrix of the reduced system: $\boldsymbol{K}_R(\boldsymbol{\theta}) = \boldsymbol{\Phi}^T \boldsymbol{K}(\boldsymbol{\theta})\boldsymbol{\Phi}$, and $\boldsymbol{Q}_R(\boldsymbol{\theta})$ is the reduced flow vector: $\boldsymbol{Q}_R(\boldsymbol{\theta}) = \boldsymbol{\Phi}^T \boldsymbol{Q}(\boldsymbol{\theta})$.

To control the error introduced by the reduced order model, one can investigate the residual error associated with the equilibrium equations considering the approximate response (see Gogu et al. [7]). That is,

$$\varepsilon(\boldsymbol{\theta}) = \frac{\|\boldsymbol{K}(\boldsymbol{\theta})\boldsymbol{H}^A(\boldsymbol{\theta}) - \boldsymbol{Q}^A(\boldsymbol{\theta})\|}{\|\boldsymbol{Q}^A(\boldsymbol{\theta})\|} \tag{10}$$

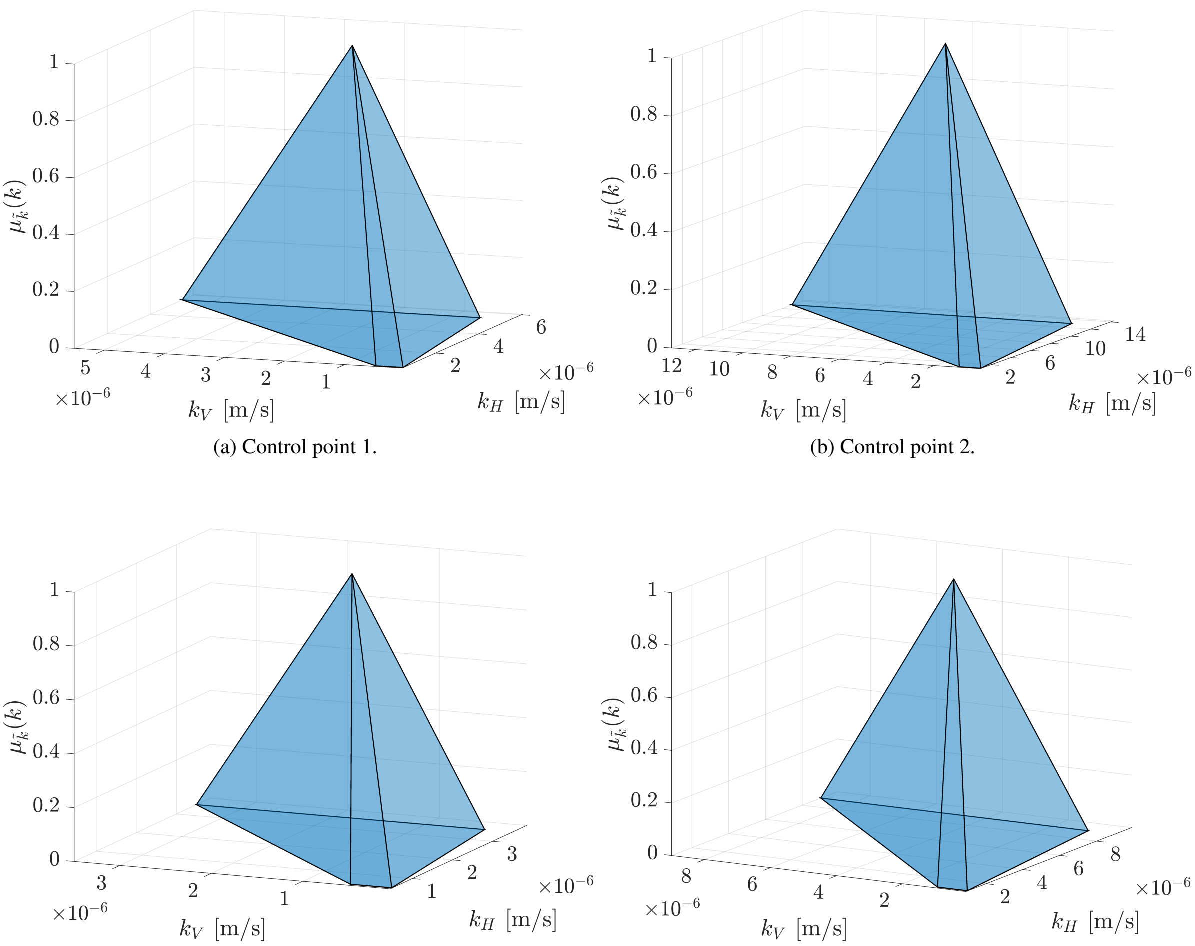
where $\varepsilon(\boldsymbol{\theta})$ is the error measure and $\|\cdot\|$ denotes Euclidean norm. The error $\varepsilon(\boldsymbol{\theta})$ is monitored at each α-cut during the optimization process (i.e. solution of eq. (4) and eq. (5)). The reduced basis was updated each time the largest error produced for one limit (for a specific membership level) exceeded a predefined defined threshold ε_t.

3 Example

The analysis of steady-state confined seepage below an impermeable dam is considered to illustrate the proposed approach. The geometrical definition of the system is based on the work of Valdebenito et al. [8]. The dam is founded on a permeable soil layer limited by an impermeable rock layer. The objective is to determine the flow that drains downstream of the dam. The dam's upstream side retains a water column of a height of 10 [m]. The soil layer has a depth of 20 [m], and its horizontal k_H and vertical k_V permeability are characterized as fuzzy fields. Four control points were considered, where the dependence between both permeabilities is included considering $0.1k_H \leq k_V \leq k_H$ (see Fanchi [9]). The membership functions associated with each control point are shown in Fig. 2.

A simple finite element model is considered, which comprises 3183 nodes and 1498 quadratic triangular elements. The system was studied considering two models: the exact model and the approximate model $R1$. The results were evaluated considering that (a) there is no basis updating process ($\varepsilon_t \to \infty$), and (b) there is a basis updating process with a threshold of $\varepsilon_t = 10^{-4}$.

Figure 3 presents the estimation of the membership function associated with the response. The results produced with the reduced basis R1 provide satisfactory match with the exact system's response. In case (a), some minimum discrepancies can be noted for low values of the membership function on the left side. These differences are not present when the procedure considers the basis updating strategy. Note that the proposed approach brings good benefits regarding to the computation time. The speedup factor associated with R1 is 45.6 for $\varepsilon_t \to \infty$ while the speedup factor associated with R1 for $\varepsilon_t = 10^{-4}$ is 35.1.



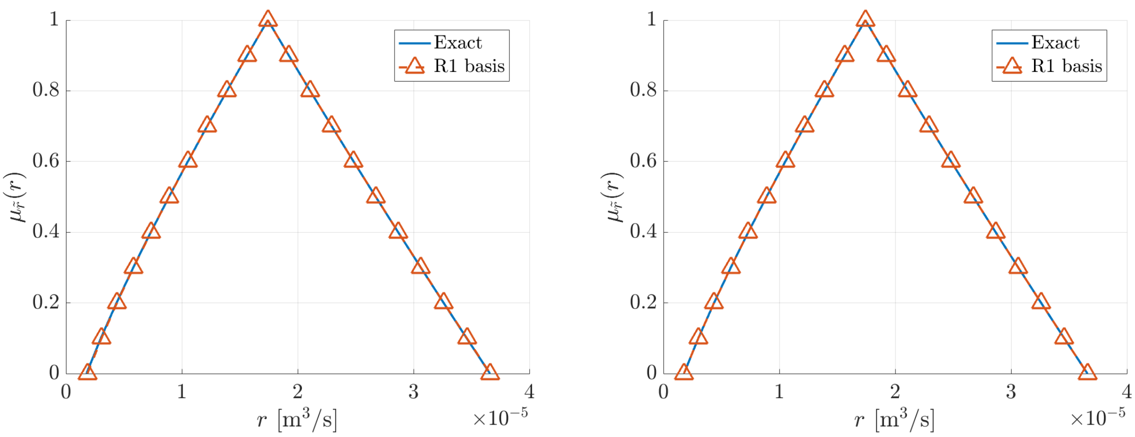
(a) Control point 1.

(b) Control point 2.

(c) Control point 3.

(d) Control point 4.

Figure 2. Membership function of permeabilities at control points. Each control point is located at: (a) $x = 30$ [m], $y = 4.5$ [m]. (b) $x = 30$ [m], $y = 15$ [m]. (c) $x = 75$ [m], $y = 2$ [m]. (d) $x = 75$ [m], $y = 9$ [m].



(a) $\varepsilon_t = \infty$

(b) $\varepsilon_t = 10^{-4}$

Figure 3. (a) Membership function associated with the response for $\varepsilon_t = \infty$. (b) Membership function associated with the response for $\varepsilon_t = 10^{-4}$.

4 Conclusions

This contribution presents a technique to estimate the fuzzy response of a seepage problem considering the spatial uncertainty in soil permeability by applying a reduced order model. The approach is formulated to propagate the uncertain permeability characterized by fuzzy fields through an optimization scheme. In particular, the subsequent challenges are addressed: (1) spatial dependencies in the horizontal and vertical permeability; and (2) the numerical cost associated with the resolution of the exact system.

The results demonstrate that a precise estimation of the fuzzy response can be obtained at reduced numerical efforts, controlling the quality of the results. Furthermore, it exhibits that fuzzy fields are a useful strategy for spatial uncertainty quantification under limited data. Nevertheless, the exhibited results should be regarded as an initial approximation of fuzzy field analysis. Forthcoming studies steps will aspire to explore more complex systems, for example, considering other types of responses and extensions to more physical dimensions.

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