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A Comparison between Dual Reciprocity and Direct Interpolation Techniques for Solving the Helmholtz Problem by Frequency Scanning

T. G. Balista[1], C. F. Loeffler Neto[1], L. O. C. Lara[1]

[1]*Mechanical Engineering Post-Graduate Program, Federal University of Espírito Santo, UFES, CT, Av. Fernando Ferrari, 540-Bairro Goiabeiras, 29075-910 Vitoria, ES, Brazil*
thiago.balista@gmail.com, loefflercarlos@gmail.com, castrolara@hotmail.com

Abstract. The identification of the modal content of a dynamical system through the excitation frequency scanning procedure is a very common procedure, especially with regard to experimental models. In terms of numerical simulation, this technique is also very accessible and computationally inexpensive. In the case of the Boundary Element Method (BEM), this procedure is much simpler than the direct solution of the associated eigenvalue problem, if the fundamental solution is frequency-dependent since the problem becomes nonlinear. In order to simplify the solution of these problems with the BEM, which are stationary acoustics problems governed by the Helmholtz equation, techniques were developed that use simpler fundamental solutions. Among these are the well-known dual reciprocity technique (DRBEM) and the more recent direct interpolation technique (DIBEM). Both are characterized by employing radial basis functions and thus avoiding domain integrations generated by the reactive term of the governing equation; however, Dual Reciprocity interpolates only the primal variable of the problem, while the Direct Interpolation technique approximates the entire kernel of the domain integral. Although they allow the direct solution of the eigenvalue problem, this article compares the two techniques mentioned to solve the problem of stationary acoustics, through scanning of imposed frequencies. The stationary data was obtained in a chosen frequency range. Error curves are obtained by comparing numerical solutions and available analytical solutions for a more accurate assessment.

Keywords: Boundary Element Method, Dual Reciprocity, Direct Interpolation.

1 Introduction

Only Great challenges have arisen with the advances in modern engineering, and so the search for solutions to engineering problems with a high level of sophistication becomes necessary. In this context, formulations related to numerical methods are in a prominent position and have evolved significantly. Several research projects have pointed out a good performance of the Boundary Element Method (BEM) in applications in which the operators that mathematically characterize the governing equation are self-adjoint [1]. However, some problems are not expressed by differential operators that have this specific property or they present a very complex inverse integral form.

Thus, the development of BEM formulations that transform domain integrals into boundary integrals using approximations with a sequence of Radial Basis Functions (RBF), has become one of the main means of arriving at this inverse form. The first strategy to solve domain action problems consists of the so-called Dual Reciprocity (DRBEM) developed by Nardini and Brebbia [2] in 1982 with the purpose of solving dynamical problems [3], and later its application was extended to other classes of problems, for example, diffusion in transient regime [4] and acoustic problems [5].

Briefly, DRBEM consists of transforming the domain action by a linear combination of a product of new functions, which can be operationalized and thus transform domain integrals using the properties of integration by parts and the Divergence Theorem. Although the results are satisfactory for some applications, the DRBEM faces some difficulties, such as requiring too many interpolations basis points (poles) to accurately represent the solution. This large number of poles results in problems of poor matrix conditioning and numerical inaccuracies.

Recently, a new alternative has been proposed that also uses RBF to solve domain action problems, called Direct Interpolation (DIBEM) [6]. Different from DRBEM, where there is the formation of two new auxiliary matrices, which are applied to the traditional boundary element matrices H and G, in DIBEM the complete kernel of the domain integral is approximated, including the fundamental solution and, consequently, presents fewer inaccuracies and numerical instabilities since the transformation to eliminate the domain integral is composed of a single matrix. Thus, the DIBEM is a technique more similar to a classical interpolation procedure. It is noteworthy that the DIBEM, although recent, has been successfully implemented in two-dimensional problems involving the solution of the Poisson equation [6] and the Helmholtz equation [7]; however, it is also still under testing through new applications, for example, the elasticity problems and the acoustic wave problems.

Thus, a frequency scanning procedure (i.e., response problems) was chosen to compare the performance of the two solution techniques based on the BEM procedure, solving two-dimensional problems, related to the Helmholtz equation. The goal with this analysis is to highlight the performance of the DIBEM technique compared to the DRBEM technique. For the example studied, linear elements are used in all formulations.

2 Governing Equation

Initially, the Helmholtz Equation can be interpreted as a simplification of the Acoustic Wave Equation [8, 9], in which the stationary amplitude $u(\boldsymbol{X})$ is produced in the system by a variable excitation whose frequency ω is known. Therefore, for a two-dimensional case, the governing equation is given:

$$u_{,ii}\,(\boldsymbol{X}) = -\frac{\omega^2}{k^2} u(\boldsymbol{X}) \tag{1}$$

It is important to emphasize that when the physical problem is expressed by means of differential equations, it becomes necessary to complement the differential equation with certain additional information, called boundary conditions, so that the physical problem is well characterized and presents a single solution. Thus, the two main types of these conditions assumed here are presented: (a) Essential Boundary Condition Γ_u (or Dirichlet's), which prescribes the basic or potential variable along the physical problem domain boundary; (b) Natural Boundary Condition Γ_q (or Neumann's), which prescribes the normal derivative of the basic variable. It is interesting to point out that the domain $\Omega(\boldsymbol{X})$ can represent a system, a body or a control volume of a given physical problem.

Thus, considering the mathematical procedures known in the context of the Boundary Element Method (BEM) [10], one has the inverse integral form associated with Eq. (1), resulting in the following equation:

$$c(\boldsymbol{\xi})u(\boldsymbol{\xi}) + \int_\Gamma u(\boldsymbol{X})q^*(\boldsymbol{\xi};\boldsymbol{X})d\Gamma - \int_\Gamma q(\boldsymbol{X})u^*(\boldsymbol{\xi};\boldsymbol{X})d\Gamma = \frac{\omega^2}{k^2}\int_\Omega u(\boldsymbol{X})\,u^*(\boldsymbol{\xi};\boldsymbol{X})d\Omega \tag{2}$$

In Eq. (2), the term $\boldsymbol{X}$ represents the field point of a specific domain $\Omega(\boldsymbol{X})$ with boundary set at $\Gamma(\boldsymbol{X})$. The prescribed source point in the system is $\boldsymbol{\xi}$; and k is the propagation velocity of acoustic waves. The term $c(\boldsymbol{\xi})$ is a coefficient that submits from the location of the source point $\boldsymbol{\xi}$ to the domain $\Omega(\boldsymbol{X})$ and, considering that it may be located on the boundary $\Gamma(\boldsymbol{X})$, the smoothness also influences [1]. It is emphasized that, $u(\boldsymbol{X})$ is the scalar potential and $q(\boldsymbol{X})$ is its normal derivative; $u^*(\boldsymbol{\xi};\boldsymbol{X})$ is the fundamental solution correlated to the Laplace problem and $q^*(\boldsymbol{\xi};\boldsymbol{X})$ is its normal derivative [1, 10]:

$$u^*(\boldsymbol{\xi};\boldsymbol{X}) = -\frac{\ln[r(\boldsymbol{\xi};\boldsymbol{X})]}{2\pi} \tag{3}$$

In Eq. (3), $r(\boldsymbol{\xi},\boldsymbol{X})$ is the Euclidean distance between the source point $\boldsymbol{\xi}$ and any point in the domain $\boldsymbol{X}$; and $\eta_i(\boldsymbol{X})$ is the outer normal on the boundary $\Gamma(\boldsymbol{X})$. Thus, the Dual Reciprocity (DRBEM) and Direct Interpolation (DIBEM) procedures make use of a fundamental solution $u^*(\boldsymbol{\xi},\boldsymbol{X})$; however, they differ in their approach to the domain integral referring to inertia in the Helmholtz Equation (i.e., the direct side of Eq. (2)). In the following sections, due to limited space, the main aspects of both formulations are presented. Therefore, the detailed development of DRBEM and DIBEM can be found in the specialized literature [2, 6].

3 Dual Reciprocity Technique

The technique of Dual Reciprocity (DRBEM) was initially developed by Nardini and Brebbia [3] for dynamical problems, and later Loeffler and Mansur [5] extended its use to model problems with domain actions, which proved to be efficient. Briefly, the first step of the DRBEM approach consists in proceeding to the following approximation for the potential present in the kernel of the domain integral of Eq. (2):

$$\mathrm{u}(\boldsymbol{X}) \approx \alpha^i F^i(\boldsymbol{X^i};\boldsymbol{X}) = \alpha^i \Psi,^i_{jj}(\boldsymbol{X^i};\boldsymbol{X}) \tag{4}$$

In Eq. (4) the coefficients α^i are initially unknown and $F^i(\boldsymbol{X^i};\boldsymbol{X})$ are the auxiliary interpolation functions that belong to the class of Radial Basis Functions (RBF) [11]. The functions $\Psi^i(\boldsymbol{X^i};\boldsymbol{X})$ are primitives of the functions of $F^i(\boldsymbol{X^i};\boldsymbol{X})$; thus, allowing the transformation of domain integrals into boundary integrals, through the aid of the methodology of integration by parts and application of the Divergence Theorem. In operational terms, it is generally interesting to choose arbitrary base points $\boldsymbol{X^i}$ in coincidence with nodal points. These points should also be located internally to improve the proposed interpolation within the domain. After applying this procedure, it is possible to write the domain integral generated by the integral formulation as follows:

$$\int_\Omega u(\boldsymbol{X})u^*(\boldsymbol{\xi};\boldsymbol{X})d\Omega = -\alpha^i\left\{c(\boldsymbol{\xi})\Psi^i(\boldsymbol{\xi}) + \int_\Gamma \left[\eta^i(\boldsymbol{X^i};\boldsymbol{X})u^*(\boldsymbol{\xi};\boldsymbol{X}) - \Psi^i(\boldsymbol{X^i};\boldsymbol{X})q^*(\boldsymbol{\xi};\boldsymbol{X})\right]d\Gamma\right\} \tag{5}$$

It is interesting to point out at this point that $\eta^i(\boldsymbol{X^i};\boldsymbol{X})$ and $\Psi^i(\boldsymbol{X^i};\boldsymbol{X})$ are known functions and are related to the function $F^i(\boldsymbol{X^i};\boldsymbol{X})$, which in turn can be chosen arbitrarily. Generally, using the mathematical procedures known in the BEM context, the field points are taken in coincidence with the source points $\boldsymbol{\xi}$. For reasons of space, the discretization procedure and the matrix treatment of this equation will not be presented. Details can be gleaned from previous work [3]. In this way, the domain integral is transformed into a matrix that corresponds to the inertia property of the system:

$$[\boldsymbol{H}]\{\boldsymbol{u}\} - [\boldsymbol{G}]\{\boldsymbol{q}\} = [\boldsymbol{H\Psi} - \boldsymbol{G\eta}]\{\boldsymbol{\alpha}\} = [\boldsymbol{H\Psi} - \boldsymbol{G\eta}]\boldsymbol{F^{-1}}\{\boldsymbol{u}\} = \frac{\omega^2}{k^2}[\boldsymbol{M}]\{\boldsymbol{u}\} \tag{6}$$

4 Direct Interpolation Technique

The Direct Interpolation (DIBEM) technique is conceptually similar to the Dual Reciprocity (DRBEM); however, the domain actions are accompanied by the fundamental solution, which depend on the source point. For this reason, the classical DIBEM presents singularity problems in the fundamental solution [7] and, as a strategy, it can be solved using the regularization procedure [12].

Recently, concerning Helmholtz problems solved by frequency scanning, a new strategy has been developed to deal with the aforementioned problem and thus avoid singularity, a new strategy has been developed to deal with the aforementioned problem and thus avoid singularity. This new strategy, so that the usual mathematical operations of the Boundary Element Method (BEM) are maintained, makes use of an auxiliary function $b^*(\boldsymbol{\xi};\boldsymbol{X})$, as indicated in Eq. (7):

$$\mathrm{b}^*(\boldsymbol{\xi};\boldsymbol{X}) = u^*(\boldsymbol{\xi};\boldsymbol{X}) - \frac{\omega^2}{k^2}G^*(\boldsymbol{\xi};\boldsymbol{X}) \tag{7}$$

The auxiliary function $b^*(\boldsymbol{\xi};\boldsymbol{X})$ is composed of two functions: the fundamental solution related to the Poisson problem added to the function $G^*(\boldsymbol{\xi};\boldsymbol{X})$, which is the Galerkin Tensor associated to $u^*(\boldsymbol{\xi};\boldsymbol{X})$, that is, its primitive [2]. By substituting Eq. (7) into the Hemholtz equation (see Eq. (1)) and, integrating it over the entire physical domain of the problem, the regularized integral of DIBEM can be written as:

$$\begin{aligned} c(\boldsymbol{\xi})u(\boldsymbol{\xi}) + \int_\Gamma [u(\boldsymbol{X})q^*(\boldsymbol{\xi};\boldsymbol{X}) - q(\boldsymbol{X})u^*(\boldsymbol{\xi};\boldsymbol{X})]d\Gamma + \frac{\omega^2}{k^2}\int_\Gamma u,_i(\boldsymbol{X})\eta_i(\boldsymbol{X})G^*(\boldsymbol{\xi};\boldsymbol{X})d\Gamma - \\ \frac{\omega^2}{k^2}\int_\Gamma u(\boldsymbol{X})G,^*_i(\boldsymbol{\xi};\boldsymbol{X})\eta_i(\boldsymbol{X})d\Gamma = -\left(\frac{\omega^2}{k^2}\right)^2\int_\Omega u(\boldsymbol{X})G^*(\boldsymbol{\xi};\boldsymbol{X})d\Omega \end{aligned} \tag{8}$$

In this way, the entire kernel of the domain integral is interpolated, as expressed in Eq. (9):

$$u(X)G^*(\boldsymbol{\xi};\boldsymbol{X}) \cong \alpha^i F^i(\boldsymbol{X^i};\boldsymbol{X}) \tag{9}$$

Similar to DRBEM, the DIBEM also uses an interpolation function $\Psi^i(\boldsymbol{X^i};\boldsymbol{X})$, primitive of $F^i(\boldsymbol{X^i};\boldsymbol{X})$. It is interesting to note at this point that the same Radial Basis Function (RBF) used in the DRBEM is retained (i.e., the thin plate radial function). After applying this procedure, it is possible to write the domain integral of the inertia term, given by the right-hand side of Eq. (8), solely in terms of an integral involving boundary variables:

$$\int_{\Omega} u(\boldsymbol{X})G^*(\boldsymbol{\xi};\boldsymbol{X})d\Omega = \int_{\Omega} \left[\alpha^i \Psi,_{jj}^i(\boldsymbol{X^i};\boldsymbol{X})\right] d\Omega = \int_{\Gamma} \left[\alpha^i \Psi,_j^i(\boldsymbol{X^i};\boldsymbol{X})\eta_i(\boldsymbol{X})\right] d\Gamma = \alpha^i \int_{\Gamma} \eta^i(\boldsymbol{X^i};\boldsymbol{X})\, d\Gamma \tag{10}$$

Due to space issues, the matrix treatment of this equation will also not be discussed, however, it resembles and can be gleaned from previous work [7, 13]. Therefore, the final system can be written as follows:

$$[\boldsymbol{H}]\{\boldsymbol{u}\} - [\boldsymbol{G}]\{\boldsymbol{q}\} - \frac{\omega^2}{k^2}[\boldsymbol{W}]\{\boldsymbol{u}\} + \frac{\omega^2}{k^2}[\boldsymbol{S}]\{\boldsymbol{q}\} = -\left(\frac{\omega^2}{k^2}\right)^2 [\boldsymbol{M}]\{\boldsymbol{u}\} \tag{11}$$

In Eq. (12), the matrices $[\boldsymbol{H}]$ and $[\boldsymbol{G}]$ are matrices arising from the integrals to the normal derivative of the fundamental solution and to the fundamental solution, respectively. The matrices $[\boldsymbol{W}]$ and $[\boldsymbol{S}]$ are respectively the matrices arising from the integration of the directional spatial derivative of the Galerkin Tensor and of the Galerkin Tensor itself. The matrix $[\boldsymbol{M}]$ corresponds to the inertia property of the system; and $\{\boldsymbol{u}\}$ and $\{\boldsymbol{q}\}$ are the vectors for the potential and its derivative.

5 Numerical Simulation: Clamped Sheet

In this section results involving the Helmholtz equation are presented and discussed to demonstrate the performance of the DIBEM, when compared to the solution of the DRBEM. The performance of the numerical solutions was evaluated by means of the Relative Percentage Error (RPE%) curve for each method, using the analytical solution as well as the available meshes (see Tab.1).

$$\text{RPE\%} = \sum_{i=1}^{n}\left(|u_a - u_n|_i \frac{100}{n|m_{ant}|}\right) \tag{12}$$

For the calculation of (RPE%), in Eq. (12), the terms u_a and u_n represent the values of the analytical potential and the calculated numerical potential at point i respectively; n corresponds to the number of degrees of freedom or calculated potentials; and m_{ant} is the largest analytical value obtained during the example scans.

Therefore, this analysis consists in solving a problem that has a physical domain $\Omega(\boldsymbol{X})$, represented by a square with side $L = 1.0$ (unit of length), subjected to essential and natural boundary conditions along its edges. It should be noted that since this is an example of a crimped elastic bar, the numerical potentials referring to this edge (i.e., crimped edge) are not accounted for. Fig. 1 illustrates the physical domain and the boundary conditions imposed for this example, as well as the geometric characteristics and the location of the adopted coordinate system. It can be seen that, in this case, the problem becomes one-dimensional when analyzed in Cartesian coordinates.

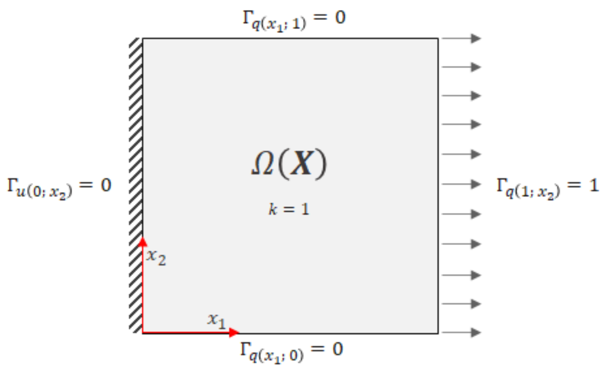

Figure 1. Clamped sheet and boundary conditions.

The value of the physical property k is considered unitary. Thus, keeping the imposed boundary conditions,

Fig. 1, the analytical solution, in terms of the potential magnitude $u(x_1)$, is given in accordance with Eq. (13):

$$u(x_1) = \sin(\omega x_1)/\cos(\omega) \tag{13}$$

It is noteworthy that in the sweep procedure, resonance values are avoided, but certain excitation frequencies are close and thereby produce error peaks in the response curve. The excitation frequencies ω used for the numerical simulations were varied from 1.00 to 20.00 in a range of 0.50. In Tab. 1, the composite meshes with linear elements for the DIBEM and DRBEM techniques are shown.

Table 1. Quantidade de pontos nodais da simulação numérica

Nomenclature	Number of Points on the Boundary	Number of Internal Points	Total Number of Points
Mesh 1		64	148
Mesh 2	84	144	228
Mesh 3		256	340
Mesh 4		144	308
Mesh 5	164	256	420
Mesh 6		484	648

Initially, in Fig. 2, the DRBEM response curves are presented for meshes with 84 and 164 elements in the boundary, respectively, and varying the internal point cloud.

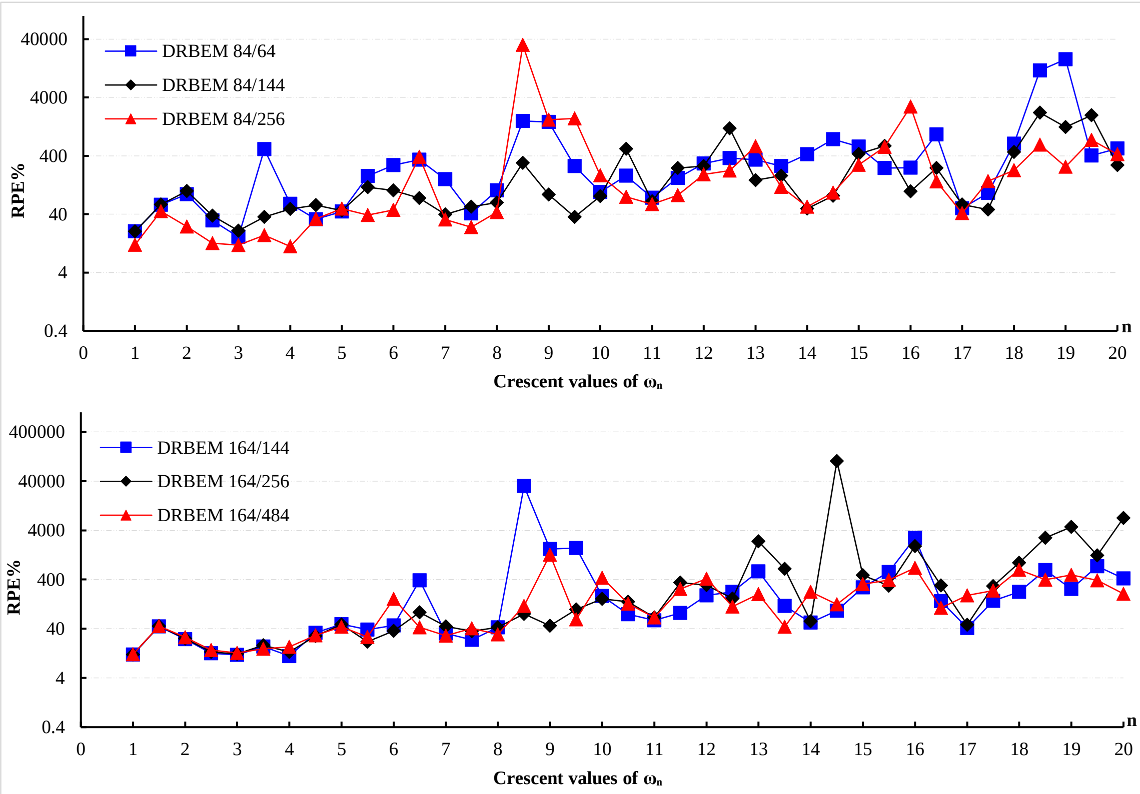

Figure 2. Comparison between RPE% curves of DRBEM for a mesh of 84 and 164 elements and different numbers of poles.

As expected, the refinement of the mesh of elements in the boundary (i.e., from 84 to 164 elements) significantly improves the quality of the results for a wider range of excitation frequencies, since these elements directly reflect in the construction of all BEM matrices. However, the introduction of a large number of internal points also produces a significant improvement in the results for higher frequencies, this is because the inertia properties must be well constituted to represent the dynamic behavior considering the higher vibration modes. It is worth noting that, excess poles, can produce inaccuracies and even divergence in the results, becoming a problem of poor matrix conditioning [7]. On the other hand, DIBEM is a technique more sensitive to the introduction of internal points and, therefore, the meshes necessarily need a larger number of these poles. This fact became even more relevant because the results obtained in numerical simulations, independent of the meshes used, are extremely superior to those obtained through DRBEM, as shown in Fig. 3.

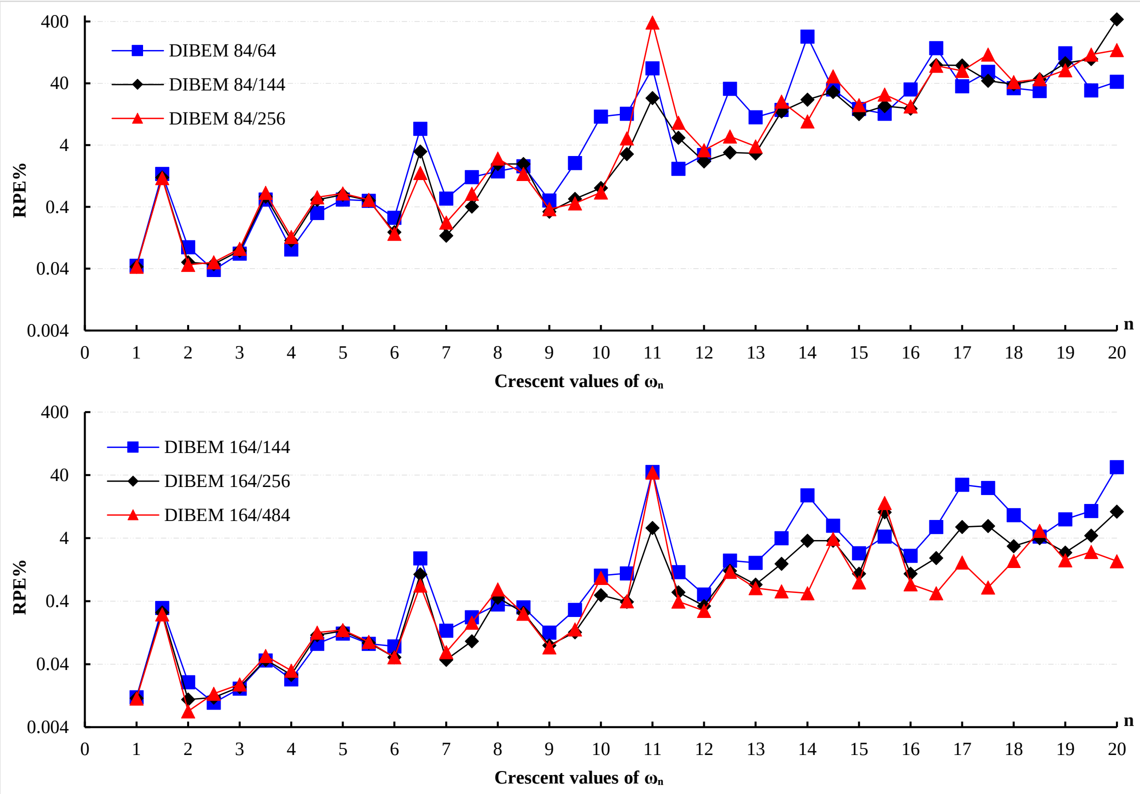

Figure 3. Comparison between RPE% curves of DIBEM for a mesh of 84 and 164 elements and different numbers of poles.

According to the simulations presented for DIBEM, a great superiority is observed in the precision of the results compared to DRBEM. The errors dropped significantly with the refinement of the mesh of elements in the boundary, because the mesh with 164 elements, obtained a better performance when compared with the mesh of 84 elements. As expected, meshing with large numbers of poles, one notices a significant improvement in the quality of the results, with a gradual drop in the trend line of the RPE% for the higher frequencies. As already mentioned, to have good accuracy in DIBEM it is necessary to have a balance between the number of points on the boundary and the number and distribution of internal points.

6 Conclusions

For many years, the DRBEM technique was the simplest alternative to overcome the mathematical difficulties that arise in applying the Boundary Element Method in problems where the operators that characterize the governing equation are not self-adjoint. However, the DIBEM has shown superior performance, robustness, and versatility for solving many scalar field cases, including the problems governed by the Helmholtz Equation.

The example solved here has regular geometry and an analytical solution available for better evaluation of the precision and convergence of the results for both techniques. It is worth mentioning that because it is a sweep of excitation frequencies, the possibility of response peaks in the results is expected. This is because the proximity between the natural frequency and the excitation frequency generates the phenomenon of acoustic resonance. In resonance detection, the DIBEM technique appeared to have greater sensitivity than DRBEM since it showed higher error peaks in numerical simulations.

The DRBEM results were not good in this example, even for low pacing rates. There was a gradual increase in the Relative Percentage (RPE%) for the high-frequency spectrum, as expected since the graphic representation of the response in terms of high frequencies implies more and more refined discretization so that the quality of the results is maintained. Ratifying the above, the RPE% curves show that the results of DIBEM are extremely superior to those of DRBEM, both with the refinement of the mesh of elements on the boundary and with the introduction of internal interpolating points.

In general, and as already mentioned, the DIBEM technique is similar to the DRBEM technique, but the basic idea of the DIBEM is to interpolate the entire kernel of the domain integral term, which includes the fundamental solution and, consequently, a larger number of internal points is critical to improving performance, compared to DRBEM. Importantly, meshes with few elements in the boundary and many internal points can generate poorly conditioned responses due to integration problems, which demands the implementation of adaptive integration procedures to avoid numerical errors.

Regarding the behavior of the DIBEM when adjusted with the auxiliary function $b^*(\boldsymbol{\xi};\boldsymbol{X})$, definitely eliminating the need for the execution of the regularization procedure, it returned coherent results in terms of accuracy, in the range of excitation frequencies analyzed.

Finally, given the parameters analyzed, in future work, it may be proposed to extend the DIBEM formulation to dynamic problems, that is, cases in which the response advances in time, in addition to performing mathematical modeling for other types of problems, for example, elasticity problems.

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