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# TWO-DIMENSIONAL ELASTIC LINEAR PROBLEMS USING THE VIRTUAL ELEMENT METHOD

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**Abstract.** Virtual Element Method (VEM) is a relatively new method for solving partial differential equations, which proposes to generalize the classical Finite Element Method (FEM) with respect to the mesh discretization. In the two-dimensional case, any simple polygon can be used as discretization element and, as a result, the shape function are not strictly polynomials. The method main characteristic is to compute those functions implicitly, without the necessity of knowing their explicit form, giving it great versatility when treating complex geometries. The VEM presents a dense mathematical formulation, and it was originally developed for the Poisson equation. In recent years, a considerable number of works has applied the method for different engineering problems usually treated using finite element models. For example, VEM formulations were adapted to different rheology problems, contact problems and topological optimization. However, the usage of the Virtual Element Method is not so popular when compared to the Finite Element Method or the Extended Finite Element Method. The main goal of this paper is to present the Virtual Element Method applied to linear elasticity problems in two-dimensions by focusing on its implementation, aiming to contribute for the popularization of the method. The results obtained with VEM are compared with the results from a Finite Element Method commercial software, showing good agreement and the great potential of the method in engineering problems.

**Keywords:** Virtual Element Method, Linear Elasticity, Finite Element Method

## 1 Introduction

The Virtual Element Method (VEM) is a relative new numerical method to solve partial differential equations. It was originally presented in the works Veiga et al. [1] and Veiga et al. [2] by a group of mathematicians. In this way, it presents a deep mathematical formulation, contemplating topics that are not usual on engineering courses.

This method proposes to generalize the classical Finite Element Method (FEM) in terms of mesh discretization. Every simple polygon can be used as discretization element in the VEM. Although, as consequence, the shape functions are not restricted to low order polynomials. The main idea of the method is to compute these function implicitly, without necessarily knowing their explicit form. This is done by projecting the function on polynomial spaces and treating the residue with a stabilization term.

The versatility of the method in terms of meshing, makes it a suitable tool for handling complex geometries. Also, in the literature, it is attested that the VEM is more robust when compared to the FEM, i.e., using fewer number of elements yields better results and, thus, lower errors. However, the method still is not very popular for classical engineering approaches, primarily due to its dense formulation. This paper main objective is to present a qualitative analysis of the VEM formulation and to present its main characteristics. And, contribute to the method popularization. Also, the method is applied to classical examples. Here, no deep stability or convergence analysis is intended, the focus is to illustrate the Virtual Element Method.

## 2 The Virtual Element Method

In this section, a qualitative overview of the VEM formulation is given. For more details of the mathematical formulation of the method, one can refer to Veiga et al. [1], Veiga et al. [3], Ortiz-Bernardin et al. [4] and Mengolini et al. [5]. Originally, the VEM was developed for the Poisson equation in two dimensions but it can be extended to solve the linear elasticity differential equations as shown in Veiga et al. [2] and Artioli et al. [6]. The method formulation is based on the hypothesis presented below.

**Hypothesis 1.** *Considering the geometric domain $\Omega$, let $h$ be the discretization parameter (e.g., polygonal diameter) and let $k \geq 1$ be a intenger called order of accuracy. For every simple polygon $K$ in the decomposition $T_h$, it holds that:*

1. *$V_h \subset H_0^1(\Omega)$, where $V_h$ is the virtual element space and $H_0^1(\Omega)$ is the closure of the continuous functions with compact support space $C_c^\infty(\Omega)$ (the sub-index c refers to compact support).*
2. *a symmetric bilinear form $a_h : V_h \times V_h \longrightarrow \mathbb{R}$ and a bilinear form $a_{h,K} : V_{h,loc} \times V_{h,loc} \longrightarrow \mathbb{R}$ such that $a_h(u,v) = \sum_{K \in T_h} a_{h,K}(u,v)$, where $V_{h,loc}$ is the local virtual element space.*
3. *a load term $f_h \in V_h'$, where $V_h'$ is the dual of $V_h$.*
4. *$\mathbb{P}_k(K) \subset V_{h,loc}$, where $\mathbb{P}_k(K)$ is the polynomial space of degree $k$ in $K$, $\mathbb{P}_{-1}(K) = \{0\}$,*
5. *k-consistency: it is true that $a_{h,K}(q,v) = a_K(q,v)$, for all $q \in \mathbb{P}_k(K)$ and for all $v_h \in V_{h,loc}$,*
6. *stability: exists constants $\alpha_1, \alpha_2 \in \mathbb{R}_+$ that are independent of the polygonal diameter $h$ and the polygon $K$ such that $\alpha_1 a_K(v,v) \leq a_{h,K}(v,v) \leq \alpha_2 a_K(v,v)$, for all $v \in V_{h,loc}$.*

With this hypothesis, it is possible to present a general overview of the method pipeline.

- **Formulation of the weak problem:** as in the Finite Element Method, the starting point to apply the Virtual Element Method is the weak formulation. To achieve the weak form, a bilinear form and a load term are defined. Also, the test functions are introduced and the function space is the closure of $C_c^\infty$ in the Sobolev space denoted by $H_0^1$. It is important to mention that the virtual element space $V_h$ is contained in the closure. In the linear elasticity context, the test functions are called virtual displacement field and the weak form is called Principle of Virtual Works.
- **Discretization of the weak form:** the discretization is made by choosing a decomposition $T_h$ of simple polygons. Before determining the discrete bilinear form and the discrete load term, it is required to prove that the discrete form has solution and define the convergence criteria. Also, stability and consistency hypothesis are required to be satisfied.
- **Construction of the virtual element space:** as mentioned before the virtual element space is a subspace of the closure. In this sense, it is possible to define local virtual element spaces and from then construct the global space. These spaces are constructed upon the definition of the degrees of freedom. The canonical choice for the degrees of freedom are the values of the functions in the vertices of each polygon, the values in each edge of each polygon and the values in the internal points (moments). In the linear case ($k = 1$), only the values on the vertices are required. It is important to mention that the choice of degrees of freedom is the same for both the Poisson equation and for the linear elasticity context. Also, to construct the the local virtual element space, a linear space $\mathbb{E}_K$ is necessary. This space is associated to the linear polynomials and to the behavior of functions in the edges of each polygon.
- **Introduction of the projection operator:** the projection operator is the responsible to compute the functions implicitly. The idea is to project functions that are not known in a first moment from the local virtual element space to a subspace of a polynomial space. In the case of the Poisson equation the projections are made directly into the polynomial spaces. In the case of linear elasticity context, there are different approaches. In Veiga et al. [2] and Mengolini et al. [5] the projections are done similarly to the Poisson equation. But in Artioli et al. [6] the projections are made directly into the strain field. In Gain et al. [7] the projection operator is divided in three parts, the first term is related to the rigid body motions, the second term is related to the constant strain modes and the third term is responsible to extract the polynomial elements. It is worth mentioning the rigid body motions and the constant strain modes are defined as subspaces of polynomial spaces.
- **Construction of the bilinear form:** the bilinear form must be constructed in order to satisfy both the consistency and stability criteria. To ensure stability, a symmetric bilinear term $S_K$ is introduced. There are different ways to define $S_K$ that can be verified in [1], Gain et al. [7], Wriggers et al. [8], Veiga et al. [9] and Artioli et al. [6]. After the choice of the stability term, the bilinear form will have a portion responsible to handle the polynomial terms and a portion responsible to handle the non-polynomial terms and high degree polynomials associated to $S_K$. Finally, the bilinear term can be computed directly from the degrees of freedom.
- **Construction of the load term:** analogous to the bilinear form, a projection operator is defined and the load term are computed directly from the degrees of freedom.
- **Construction of the stiffness matrix and load term:** to build the bilinear form matrix, one should choose an adequate basis for the virtual element space (e.g. Lagrange polynomials) and an adequate basis for the polynomial spaces and its subspaces. After the choice is made, by applying the definition of the bilinear term and the load term within the basis, one should obtain, respectively, the stiffness matrix and the load

vectors. Then, it is possible to assembly the global problem and solve it as in FEM.

## 3 Results

In this section some results regarding the application of the Virtual Element Method applied to two-dimensional linear elasticity are presented. The first application consists on the analysis of the displacement field in a unitary square plate. The numerical results are compared to the analytical ones. The second example, concerns to the analysis of the displacement field in a non-convex pentagon. The numerical results are compared with the commercial software *Ansys*. The last application is related to the classical problem of a rectangular plate with a hole. For all the applications, the plane stress state hypothesis is considered.

The *Gmsh* software was used here (for more details refers to Geuzaine and Remacle [10]) to generate meshes for the geometries used in this section. For the presented analysis only square and triangle elements are considered. The used meshes were not regular nor uniform. The Virtual Element Method implementation was mostly inspired on the works of Gain et al. [7], Sutton [11] and Ortiz-Bernardin et al. [4].

### 3.1 Unitary square plate

An unitary square plate was considered with movement restricted in the vertical direction in the bottom edge and in the horizontal direction in left edge. The thickness of the plate is considerably small when compared to the edges dimensions. Also, a distributed load $g = 1kN/m$ is applied in right edge as shown in Figure 1. The analytical solution is given, accordingly to Artioli et al. [6], by

$$u(x, y) = \frac{g}{E}x \tag{1}$$

and

$$v(x, y) = -\frac{\nu g}{E}y, \tag{2}$$

where $E = 1Mpa$ is the elastic modulus and $\nu = 0.3$ is the Poisson coefficient.

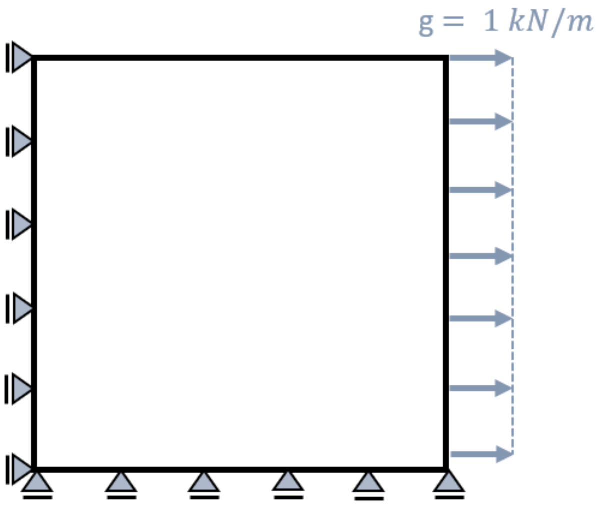


Figure 1. Unitary square plate problem.

Figure 2 shows the error of the displacement field associated with different element sizes. The error is calculated accordingly to Artioli et al. [6] and it is given by:

$$e(u_h, u) = \sqrt{\frac{\sum\limits_{\mathbf{x} \in T_h} \|\mathbf{u}(\mathbf{x}) - \mathbf{u_h}(\mathbf{x})\|^2}{\sum\limits_{\mathbf{x} \in T_h} \|\mathbf{u}(\mathbf{x})\|^2}}, \tag{3}$$

It is possible to observe the effects regarding the choice of the stabilization term given by the convergence rate that is not as expected. Also, the errors are very close independent of the size of the elements, showing the robustness of the method as discussed on Veiga et al. [9].

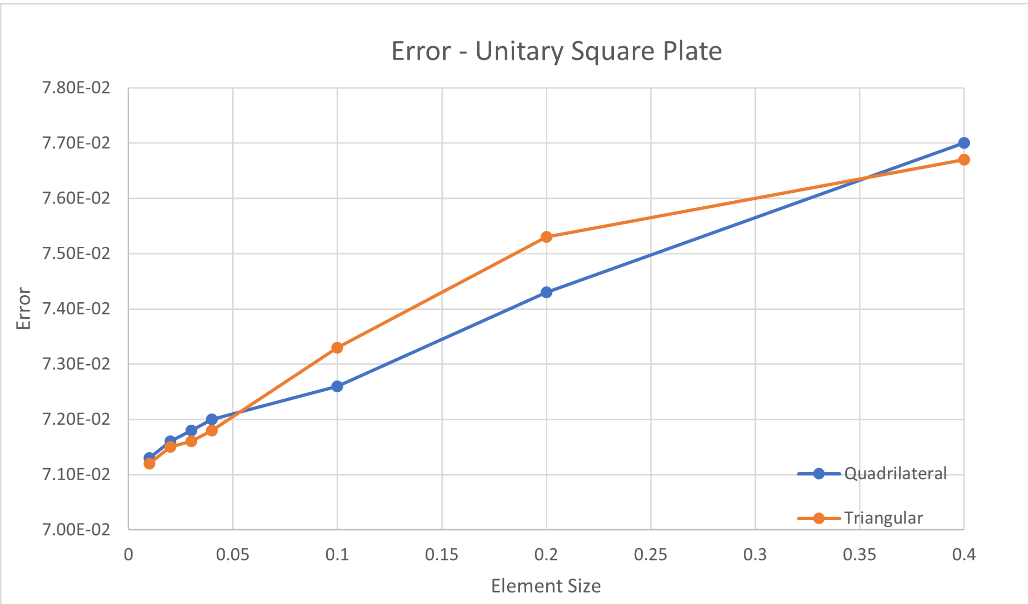


Figure 2. Unitary square plate displacement error.

### 3.2 Non-convex pentagon

For the pentagon, the same material parameters were used as for the unitary square and the problem geometry can be observed in Figure 3. Similar to the square plate, the pentagon orthogonal edges are unitary and the inclined edge has length $\sqrt{2}/2$. Figures 4, 5 and 6 show respectively the maximum absolute values for the displacements $u_h$, $v_h$ and $U$ using the VEM and *Ansys*. It is worth mentioning that $U$ refers to the total displacement given by $U = \sqrt{u_h^2 + v_h^2}$. It can be observed that the VEM convergence rate is slower than *Ansys*. Again, this might occur due to the choice of the stabilization term.

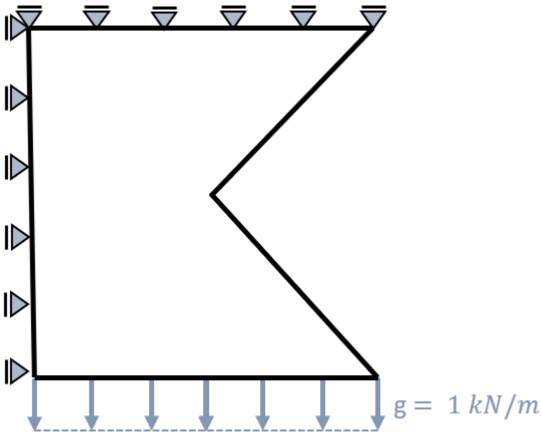


Figure 3. Non-convex pentagon problem.

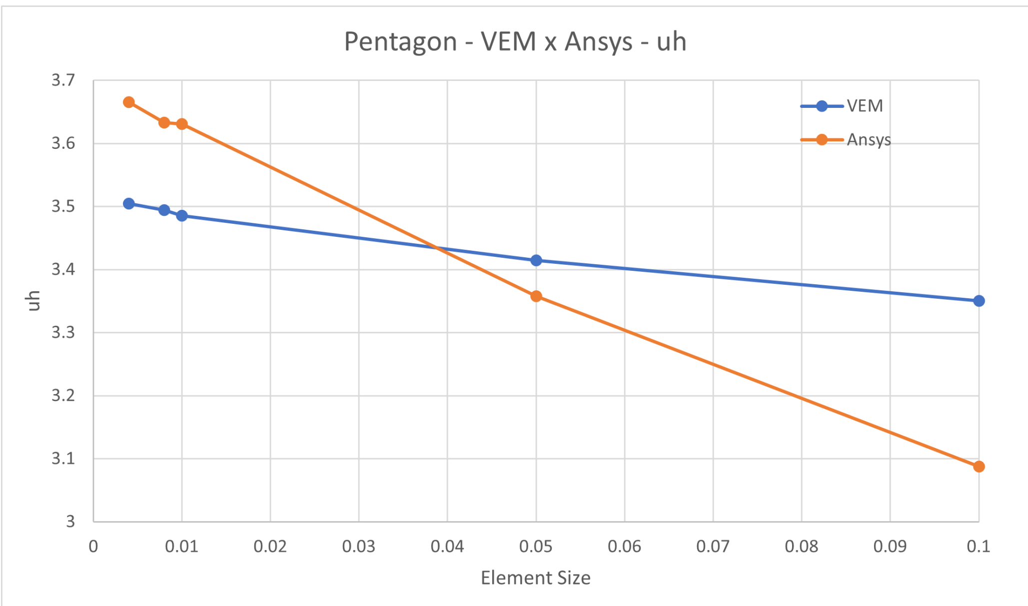


Figure 4. VEM vs. Ansys $u_h$ solution.

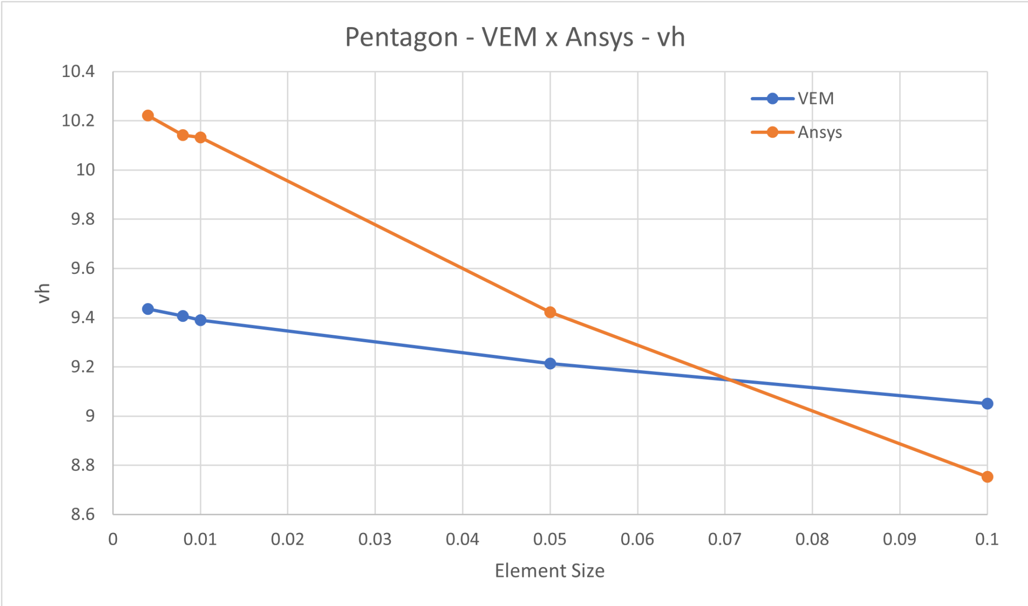


Figure 5. VEM vs. Ansys $v_h$ solution.

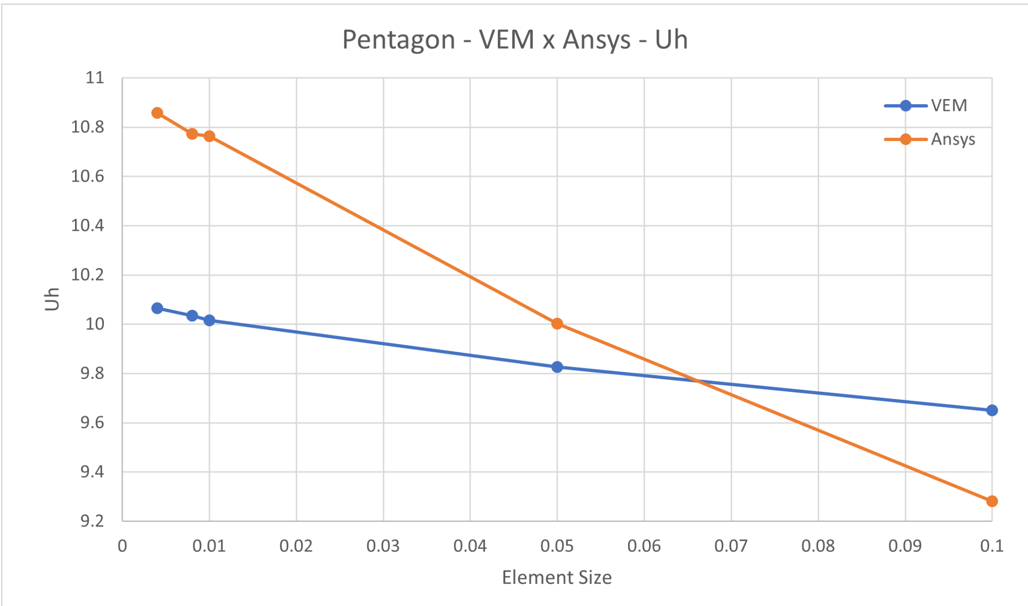


Figure 6. VEM vs. Ansys $U$ solution.

### 3.3 Plate with a hole

This application concerns to a thin rectangular plate with a hole as shown in Figure 7. The plate has width of $60cm$, height $10cm$ and thickness of $1cm$. The hole diameter is $1cm$ and a stress field $\sigma_N = 1000kN/m^2$ is applied in the edges. The main goal here is to use the Virtual Element Method to estimate the stress field and calculate the concentration factor.

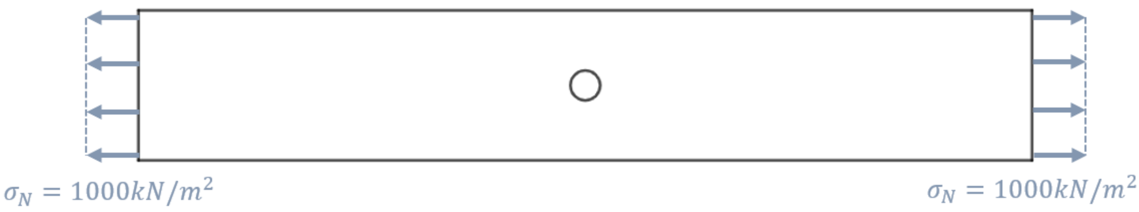


Figure 7. Plate with a hole problem.

The concentration factor is calculated as in Young and Budynas [12] and its numerical value for this geometry is given by $CF = 3.034$. Figures 8 and 9 show, respectively, the concentration factor estimated using the VEM and the error between the numerical and the analytical solution.

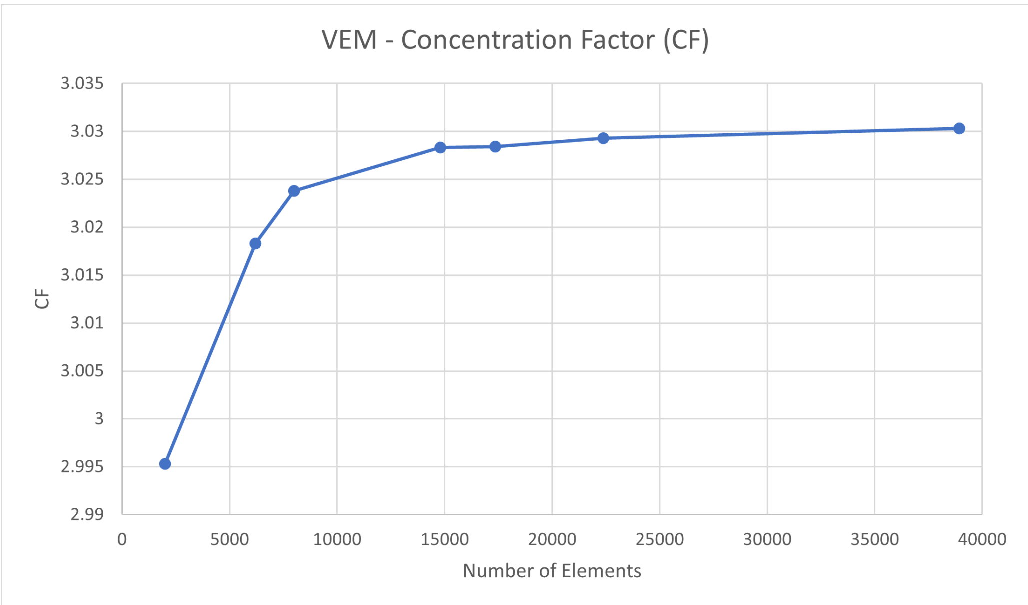


Figure 8. Concentration factor estimated with the VEM.

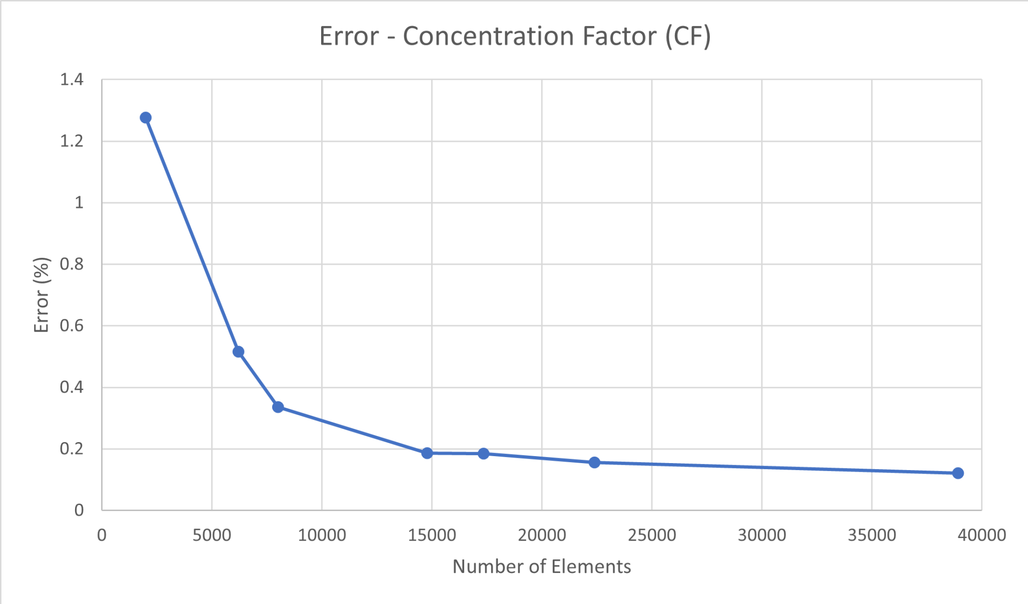


Figure 9. Error between numerical and analytical solution.

As can be seen, the error is considerably small even for few elements. For example, for 1995 elements the error is approximately $1\%$. Again, it is possible to observe the effects of the stabilization term on the convergence rate of the error.

## 4 Conclusions

In this work, a qualitative analysis of the Virtual Element Method applied to solve the differential equations in the linear elasticity context was presented. An implementation pipeline was proposed and the main characteristics of the method was presented. Three applications of the method were shown and the implementation was based on matrix frameworks that are closer to an engineering approach. Also, the implementation was restricted to the linear case ($k = 1$).

It was possible to observe that the stabilization term takes an important role in the convergence rate of the method. This occurs because the term is related to the non-polynomial terms projection residues and some stabilization terms might not represent the behavior of the non-polynomial part in an adequate. On the other hand, the method presented robustness in the sense that for few number of elements, considerably small errors were obtained.

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