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Numerical Resolution of the One-dimensional Shallow Water Equations by the Discontinuous Galerkin Method

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Abstract. The 1D shallow water equations model the unstable flow of an incompressible newtonian fluid in a channel. These equations are given by the conservation of mass and linear momentum, and can be viewed as an average of the Navier-Stokes equations under the assumption that the vertical length scale is much smaller than the horizontal length scale. These equations, also known as Saint-Venant equations, compose a system of conservation laws of hyperbolic nature, allowing for discontinuous solutions. In order to provide a numerical methodology for the approximation of the Saint-Venant equations capable of accurately representing such discontinuous solutions, in this work we adopt the Discontinuous Galerkin method. This class of finite element methods is based on the weak formulation of the differential equation to be studied, and uses discontinuous piecewise polynomial approximations. We present some numerical experiments for different flow regimes and non-horizontal beds for the 1D shallow water problem, such as idealized dam breaking, hydraulic jumps and transcritical flows in channels with non-constant bathymetry.

Keywords: Finite Element Method, Hyperbolic Systems, Shallow water equations, Discontinuous Galerkin, Saint-Venant equations.

1 Introduction

The one-dimensional shallow water equations, models the unstable flux of a newtonian fluid in open channels. These equations are given by the conservation of mass, the balance of the linear momentum and can be seen like an average of the Navier-Stokes equations. In this case, the vertical length scale of the water depth is relatively small in relation to the horizontal scales. There are usually called of the Saint-Venant equations, name that is due to the french engineer Adhémar Jean Claude Barré de Saint-Venant, who firstly published these equations in 1871 De St Venant [1].

For a channel with length $L > 0$ and a time interval $(0, T)$, the Saint-Venant equations are given by the mass balance of the fluid

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0, \quad (x,t) \in (0,L) \times (0,T) \tag{1}$$

and the linear momentum balance equation which can be written as

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A} + gI_1\right) = gA\left(S_0 - S_f\right) + gI_2, \quad (x,t) \in (0,L) \times (0,T), \tag{2}$$

where $A = A(x,t)$ is the wetted cross-section area at position x and time t, $Q = Q(x,t)$ is the flow rate, g is the gravitational acceleration (taken as 9.81 $\mathrm{m/s^2}$), I_1 and I_2 are terms that account for the hydrostatic pressure force and the wall pressure force, respectively, being expressed by

$$I_1 = \int_0^{h(x,t)} (h-y)b(x,y)\mathrm{d}y, \quad \text{and} \quad I_2 = \int_0^{h(x,t)} (h-y)\frac{\partial b(x,y)}{\partial x}\mathrm{d}y, \tag{3}$$

and where $b = b(x,y)$ is the channel width which, for a given point x, is also a function of the depth y, as depicted in Figure 1, and $h = h(x,t)$ is the water depth, which can be post-processed from A and b. Finally, $z_b(x)$ is the channel bed elevation at point x, measured from a datum, S_0 represents its spatial derivative

$$S_0 = \frac{\partial z_b}{\partial x} \tag{4}$$

and S_f designates a friction slope term that can be written as

$$S_f = \frac{n^2 Q|Q|}{R^{4/3} A^2} \tag{5}$$

where R is the hydraulic radius (ratio between the wetted area and the wetted perimeter of the channel) and n $(\mathrm{s/m}^{1/3})$ is the Manning roughness coefficient. The estimation of the Manning coefficient is a hard problem related with the simulation of floods in natural channels Lai and Khan [2].

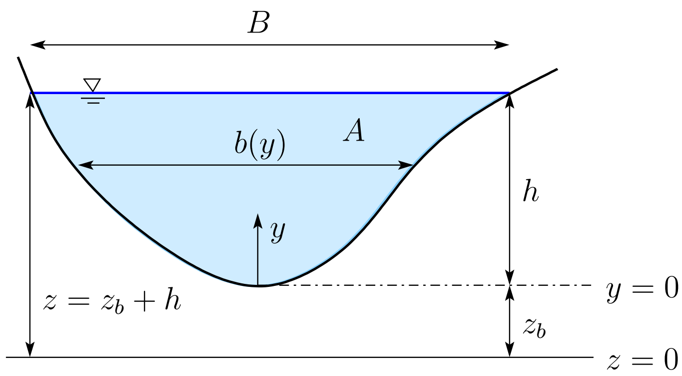

Figure 1. Illustration of a cross-section in a point $x \in (0, L)$ that shows the setting of the problem. The term B, that not appears in the equations (1) and (2), represents the maximum width of the cross-section and $b(y)$ represents the variation of the width of the cross-section.

2 One-dimensional Shallow Water Equations in Uniform Rectangular Channels

Consider now a rectangular uniform channel, that is, for any cross-section in $x \in (0, L)$, we have a rectangle with fix width $b > 0$. In this case, the Saint-Venant equations are given by

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0, \quad (x,t) \in (0,L) \times (0,T); \tag{6}$$

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q^2}{h} + \frac{gh^2}{2}\right) = gh\left(S_0 - S_f\right), \quad (x,t) \in (0,L) \times (0,T). \tag{7}$$

In the equations (6) and (7), the height of the water h and the flow rate q are the searched variables and the remaining terms are the same of the equations (1), (2) and (4), with the exception of S_f, which now is given by $S_f = \dfrac{n^2 q|q|}{h^{10/3}}$. The equations (6) and (7) are an simplification of the equations (1) and (2).

2.1 Flow Regimes

In our numerical experiments, we will simulate different flow regimes. These patterns depend on the fluid velocity and are featured by a dimensionless constant called Froude number. The Froude number $F_R(x,t)$ calculated at a point $(x,t) \in (0,L) \times (0,T)$ is equal to the quocient of the inertial and gravitational forces. It is given by

$$F_R(x,t) = \frac{u(x,t)}{\sqrt{gh(x,t)}}, \quad \text{when} \quad u(x,t) = \frac{q(x,t)}{h(x,t)}. \tag{8}$$

Using the Froude number, we have the following classification of a flow in a point $(x,t) \in (0,L) \times (0,T)$. If

$$F_R(x,t) = \begin{cases} 1, & \text{then the flow regime is critical in } (x,t) \in (0,L) \times (0,T); \\ < 1, & \text{then the flow regime is subcritical in } (x,t) \in (0,L) \times (0,T); \\ > 1, & \text{then the flow regime is supercritical in } (x,t) \in (0,L) \times (0,T). \end{cases} \tag{9}$$

Finally, a fluid is said to be in a transcritical flow regime if there is a change in the Froude number so that $F_R(x,t) > 1$ for some $x \in (0,L)$ and $F_R(y,t) > 1$ for some $y \in (0,L)$ with $x \neq y$.

3 Discontinuous Galerkin Method

We can write the equations (6) and (7) according to the vector balance law

$$\begin{aligned} \boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x &= \boldsymbol{s}(\boldsymbol{u}(x,t),x,t), \quad \text{in } (0,L)\times(0,T), \quad (10)\\ \boldsymbol{u}(x,0) &= g(x), \quad \text{for all } x\in(0,L), \quad (11) \end{aligned}$$

provided with suitable boundary conditions and where

$$\boldsymbol{u} = \begin{bmatrix} h \\ q \end{bmatrix}, \ \boldsymbol{f}(\boldsymbol{u}) = \begin{bmatrix} q \\ \dfrac{q^2}{h} + \dfrac{gh^2}{2} \end{bmatrix}, \ \boldsymbol{s}(\boldsymbol{u},x,t) = \begin{bmatrix} 0 \\ gh\,(S_0 - S_f) \end{bmatrix}. \quad (12)$$

Let $\{x_{j-1/2}, x_{j+1/2}\}_{j=1}^{N}$ a partition of the interval $(0,L)$ with intervals of the partition $I_j = (x_{j-1/2}, x_{j+1/2})$, and length $\Delta x_j = x_{j+1/2} - x_{j-1/2}$. Each interval I_j will be a element of our spatial domain $(0,L)$. We will approximate the exact solution by a numerical solution in each element I_j of the domain. This numerical solution on each interval I_j will belongs to a finite-dimensional subspace of $L^2(0,L)$, which we call the approximation space V_h.

The finite-dimensional subspace of $L^2(0,L)$ used in the method is the space of polynomials of degree up to k

$$V_h := V_h^k := \{v \in L^2(0,L); v\big|_{I_j} \in P^k(I_j),\ j = 1,2,\dots,N\}. \quad (13)$$

The local variational formulation of the problem (10)-(11) it is given by

$$\begin{aligned} &\int_{I_j} \frac{\partial \boldsymbol{u}_h}{\partial t}(x,t)\boldsymbol{v}_h(x)\,\mathrm{d}x - \int_{I_j} \boldsymbol{f}(\boldsymbol{u}_h(x,t))\frac{\partial \boldsymbol{v}_h}{\partial x}(x,t)\,\mathrm{d}x \\ &+ \boldsymbol{f}(\boldsymbol{u}_h(x_{j+1/2},t))\boldsymbol{v}_h(x_{j+1/2}^-) - \boldsymbol{f}(\boldsymbol{u}_h(x_{j-1/2},t))\boldsymbol{v}_h(x_{j-1/2}^+) = \int_{I_j} \boldsymbol{s}(\boldsymbol{u}_h(x,t),x,t)\boldsymbol{v}_h(x)\,\mathrm{d}x, \end{aligned} \quad (14)$$

$$\int_{I_j} \boldsymbol{u}_h(x,0)\,dx = \int_{I_j} g(x)\boldsymbol{v}_h(x)\,\mathrm{d}x. \quad (15)$$

The values $\boldsymbol{f}(\boldsymbol{u}_h(x_{j+1/2},t))$ and $\boldsymbol{f}(\boldsymbol{u}_h(x_{j-1/2},t))$ not are well defined, then are replaced by a numerical flux. The numerical flux that we will use in our numerical experiments will be the HLL flux. Following Khan and Lai [3],applying to the one-dimensional shallow water equations, the HLL flux it is given by

$$\boldsymbol{f}^{HLL} = \begin{cases} \boldsymbol{f}^-, & \text{if } S_L \geq 0; \\ \dfrac{S_R\boldsymbol{f}^- - S_L\boldsymbol{f}^+ + S_L S_R(\boldsymbol{u}^+ - \boldsymbol{u}^-)}{S_R - S_L}, & \text{if } S_L < 0 < S_R; \\ \boldsymbol{f}^+, & \text{if } S_R \leq 0. \end{cases} \quad (16)$$

$$\begin{aligned} S_L &= \min\left(u^- - \sqrt{gh^-}, u^+ - \sqrt{gh^+}\right); \\ S_R &= \max\left(u^- + \sqrt{gh^-}, u^+ + \sqrt{gh^+}\right); \end{aligned} \quad (17)$$

or

$$\begin{aligned} S_L &= \min\left(u^- - \sqrt{gh^-}, u^* - c^*\right); \\ S_R &= \max\left(u^- + \sqrt{gh^-}, u^* + c^*\right); \end{aligned} \quad (18)$$

when

$$u = \frac{q}{h}; \quad u^* = \frac{1}{2}(u^- + u^+) + \sqrt{gh^-} - \sqrt{gh^+}; \quad c^* = \frac{1}{2}\left(\sqrt{gh^-} + \sqrt{gh^+}\right) + \frac{1}{4}(u^- - u^+). \quad (19)$$

We want to get an approximate solution $\boldsymbol{u}_h = [u_{h,1}, u_{h,2}]^T$, when $u_{h,1}, u_{h,2} \in V_h$ and such that the variational formulation holds. The local variational formulation (14)-(15) generates a system of coupled ODE's that can be solved by a scheme for time evolution. The scheme used will be the Strong Stability Preserving Runge-Kutta (SSP-RK). Another problem to be treated is the appearance of spurious ascillations when we choose the basis of V_h, being formed by polynomials of degree greather than or equal to 1, which is predicted by the Godunov's theorem. For more information, see Cockburn and Shu [4].

We will denoted by DG0 the discontinuous Galerkin method using the approximation space with piecewise constant polynomials and by DG1 the approximation space with piecewise linear polynomials.

4 Numerical Experiments

In this section we will present some numerical experiments of the proposed discontinuous Galerkin method, applied to the shallow water equations on a uniform rectangular channel (6)-(7). The numerical experiments will be the dam-break with friction on the bed, the hydraulic jump and the flow in the transcritical regime in a domain with bumps. In the methods DG0 and DG1, we will use the HLL numerical flux. The Courant number chosen will be $\nu = 0.1$. The numerical solutions obtained by the methods DG0, DG1 in this setting and in meshes with 50 or 100 elements will be compared with the reference solutions obtained by the DG1 method on a mesh with 1000 elements and using the HLL numerical flux.

4.1 Dam-break in a rectangular channel with bed friction

We will simulate a dam-break in a horizontal rectangular flume with friction on the bed. The flume has $0.096\,\mathrm{m}$ of width, $0.08\mathrm{m}$ of heigth and $20\,\mathrm{m}$ of length, with a dam localized in $10\,\mathrm{m}$. This flume is made of wood, and the Manning roughness coefficient in this case is given by $n = 0.009\,\mathrm{s/m}^{1/3}$. The depth of water on the upstream is $0.074\,\mathrm{m}$, with dry bed on the downstream. The dam is taken away instantaneously, and the flow is simulated. The water depth and the flow rate simulated after $9.4\,\mathrm{s}$ after the dam-break are shown in the Figure 2.

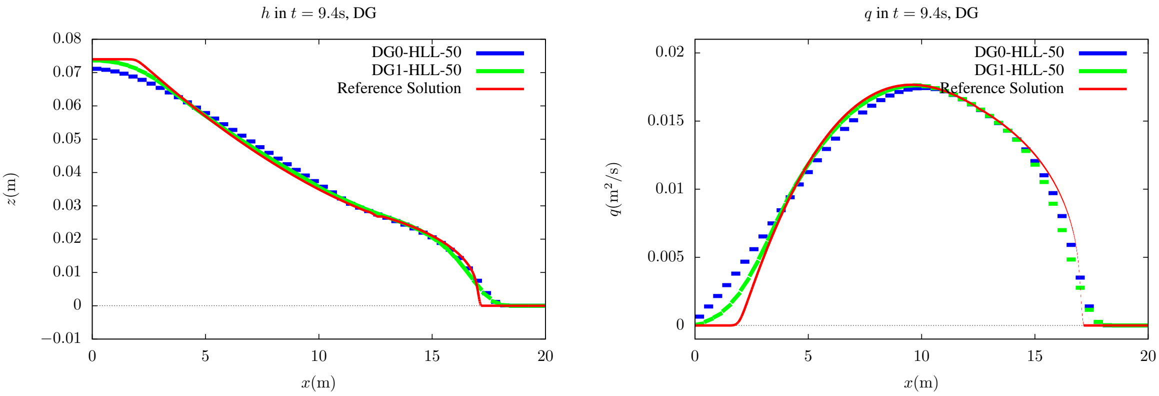

Figure 2. Simulation of a dam-break with bed friction in $t = 9.4$s by the DG method. In the left, we have the simulated water depth $z = z_b + h$. In the right, we have the simulated flow rate q.

In the Figure 2, we see that the discontinuous Galerkin provides a good approximation of the water depth compared with the reference solution and treated well the dry bed condition, which is the condition such that $h(x,t) = 0$ for some $(x,t) \in (0,L) \times (0,T)$. It can be seen that the DG1 method approximates the reference solution better than the DG0 method. We also see that the DG method approximates the flow rate without spurious oscillations.

4.2 Hydraulic jump

We will run the simulation of a hydraulic jump. The channel has $14\,\mathrm{m}$ of length and $0.46\,\mathrm{m}$ of width, with horizontal bed. The Manning roughness coefficient is taken equal to $n = 0.008\,\mathrm{s/m}^{1/3}$. The water depth in the initial condition is $0.031\,\mathrm{m}$, and a discharge of $0.118\,\mathrm{m}^2/\mathrm{s}$ are specified. On the downstream, the water depth grows from $0.031\,\mathrm{m}$ to $0.265\,\mathrm{m}$ in $50\,\mathrm{s}$, and remains constant equal to $0.265\,\mathrm{m}$ after that. The numerical solutions in steady state for the water depth and the flow rate are shown in the Figure 3.

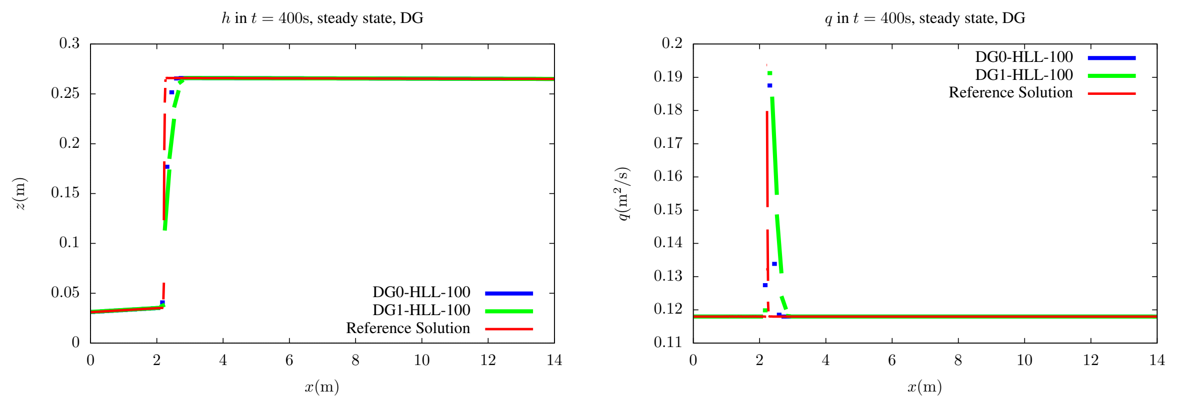

Figure 3. Simulation of a hydraulic jump in steady state. On the left, we have the simulated water depth $z = z_b + h$. On the right, we have the simulated flow rate q.

In the Figure 3, we see that the DG1 method fails in the exact location of the shock compared with the reference solution. In this experiment, the DG0 method better captures the shock location and the formation of discontinuities. We observe that the failure is generated by the lack of balance of the shallow water equations in the DG method. As the solution is in steady state, the flow rate should be constant, which does not happen in the DG0 method, in the DG1 method and neither in the reference solution, that is given by the DG1 method in a super-refined mesh.

4.3 Flow in the transcritical regime in a domain with bumps

We have a channel without friction and with $1\,\mathrm{m}$ of width and $25\,\mathrm{m}$ of length, with bed elevation z_b given by

$$z_b(x) = \begin{cases} 0.2 - 0.05(x-10)^2, & \text{if } 8 \leq x \leq 12; \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

We will simulate a fluid with transcritical flow regime on a channel with bumps, using the initial water depth given by $0.33\,\mathrm{m}$, with dry bed. The flow rate on the upstream is given by $0.18\,\mathrm{m}^2/\mathrm{s}$, and the water depth on the downstream it is $0.33\,\mathrm{m}$. The fluid changes from the subcritical to the supercritical regime, and then back to the subcritical regime along the x-axis through a hydraulic jump. The numerical solutions in steady state are shown in the Figure 4.

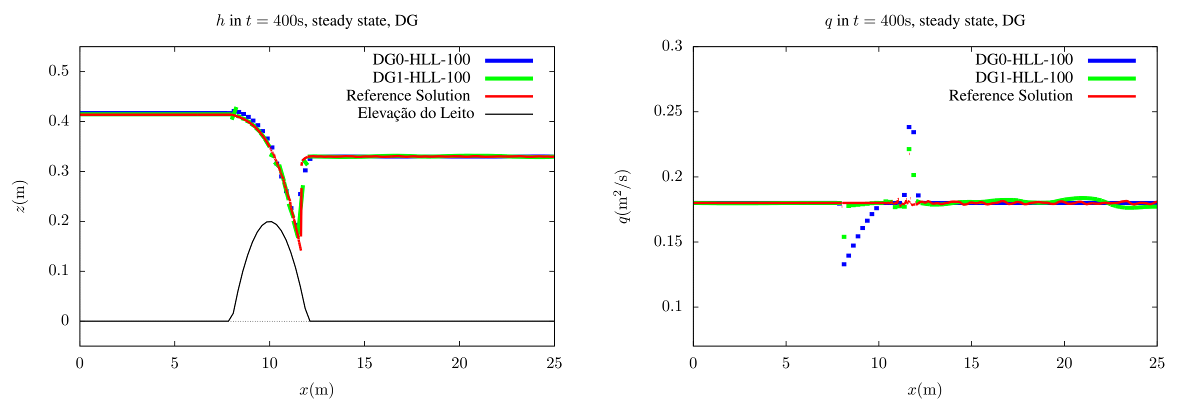

Figure 4. Simulation of a transcritical flow by the DG method in steady state. On the left, we have the simulated water depth $z = z_b + h$. On the right, we have the simulated flow rate q.

From Figure 4, we see that the DG method can approximate the reference solution well. The approximation of the flow rate, again, it is not good. As we are in the steady state, we should have constant flow rate, but the lack of balance of the equations generates spurious oscillations in the numerical approximation of the flow rate by the DG method.

5 Conclusions

The discontinuous Galerkin method it is a appropriate choice for solving the one-dimensional shallow water problem, because it is able to capture shock waves and allows the increase of the accuracy order with the use of slope limiters. However, the application of the method in balance laws, where source terms are presented, requires a certain balance of the discrete equations, in order to approximate the solution well near the discontinuities and regions where the bed slope is not smooth. For future work, it is proposed to study and implement the well-balanced methodology of Xing [5] and Lai and Khan [2], for the proper treatment of this problem.

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