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# On the application of a global post-processing strategy for stress recovery in nearly-incompressible elasticity problems

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**Abstract.** In the context of linear elasticity problems, the two major variables to be determined are the displacement and the stress tensor fields, which are often required in real-world applications. However, some classical and widely used finite element methods for these problems only provide approximations for the displacement, and the stress tensor needs to be post-processed. In this work, we study a post-processing strategy for the stress recovery obtained by combining the weak form of the constitutive equation with a least-square residual of the equilibrium equation. We will focus our studies on the application of this post-processing strategy in elasticity problems with nearly-incompressible materials, which are known in the literature to offer additional challenges. Providing an accurate approximation for the displacement field, our main goal is to evaluate whether the post-processing strategy is able to obtain satisfactory approximations for the stress tensor field on nearly-incompressible problems.

**Keywords:** Finite element methods, Linear elasticity, Post-processing strategies, Nearly-incompressible problems

## 1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain occupied by an elastic material. Given a load function $\boldsymbol{f} \in L^2(\Omega, \mathbb{R}^2)$, a fourth-order elasticity tensor $\boldsymbol{C}$, and suitable boundary conditions, the linear elasticity problem (in two dimensions) consists of finding the displacement vector $\boldsymbol{u}$ such that

$$\operatorname{div}(\boldsymbol{C}\boldsymbol{\varepsilon}(\boldsymbol{u})) = \boldsymbol{f} \quad \text{in} \quad \Omega, \tag{1a}$$

$$\boldsymbol{u} = 0 \quad \text{on} \quad \partial\Omega, \tag{1b}$$

where $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the symmetric part of the gradient of $\boldsymbol{u}$ and the divergence operator is defined row-wise. In (1b) we set homogeneous Dirichlet boundary conditions, which is be enough for our studies, but more general conditions could also be considered.

For the particular case of homogeneous and isotropic materials, the elasticity tensor $\boldsymbol{C}$ can be characterized by the Lamé coefficients $\lambda \geq 0$ and $\mu > 0$ according to

$$\boldsymbol{C}\boldsymbol{S} = 2\mu\boldsymbol{S} + \lambda \operatorname{tr}(\boldsymbol{S})\boldsymbol{I}, \quad \forall \boldsymbol{S} \in \mathbb{S}, \tag{2}$$

with $\mathbb{S}$ being the space of symmetric second order real tensors, $\boldsymbol{I}$ the identity tensor and $\operatorname{tr}$ the trace operator. The tensor $\boldsymbol{C}$ defined by (2) is uniformly positive definite and therefore invertible with the inverse, also known as compliance tensor, given by

$$\boldsymbol{A}\boldsymbol{S} = \frac{1}{2\mu}\left(\boldsymbol{S} - \frac{\lambda}{2(\lambda+\mu)} \operatorname{tr}(\boldsymbol{S})\boldsymbol{I}\right), \quad \forall \boldsymbol{S} \in \mathbb{S}. \tag{3}$$

When the Lamé coefficient $\lambda$ tends to infinity, the corresponded elasticity problem is said to be nearly-incompressible or near the incompressibility limit. From the numerical perspective, nearly-incompressible problems offer additional challenges in their approximation, and many classical methods, such as the continuous Galerkin method, do not perform well [1, 2].

In elasticity problems, it is often helpful to rewrite the second-order differential equation (1a) as a first order system by introducing a new variable $\boldsymbol{\sigma}$

$$\boldsymbol{A}\boldsymbol{\sigma} = \varepsilon(\boldsymbol{u}) \quad \text{in} \quad \Omega, \tag{4a}$$

$$\operatorname{div}\boldsymbol{\sigma} = \boldsymbol{f} \quad \text{in} \quad \Omega, \tag{4b}$$

where equations (4a) and (4b) are referred respectively as the constitutive equation and the balance equation. The variable $\boldsymbol{\sigma}$ is called stress tensor or stress field, and many applications require approximations for this variable, in addition to the displacement $\boldsymbol{u}$ [3, 4]. Some finite element methods, such as the mixed displacement-stress methods, can approximated both variables simultaneously [3, 5]. Other methods however only provide approximations for the displacement and the stress field needs to be approximated by an auxiliary post-processing strategy. Once computed an approximation $\boldsymbol{u}_h$ for the displacement, the most simple post-processing strategy for the stress consists in defining the approximated stress element-wise through the constitutive equation

$$\tilde{\boldsymbol{\sigma}}_h|_K = \boldsymbol{C}\varepsilon(\boldsymbol{u}_h|_K), \tag{5}$$

for each element $K$ of a domain discretization $\mathcal{T}_{\mathrm{h}}$.

This strategy, although simple and with almost no extra computational cost, is not always recommended. In general the approximated stress $\tilde{\boldsymbol{\sigma}}_h$ does not belong to the space $H(\operatorname{div}, \Omega, \mathbb{M})$, which may be a problem depending on the application. Another problem, arguable more serious, appears in nearly-incompresssible problems, where even if the displacement approximation $\boldsymbol{u}_h$ is accurate, the strategy (5) does not guarantee the accuracy of the approximated stress $\tilde{\boldsymbol{\sigma}}_h$, and often the absolute errors found are prohibitive large.

For those reasons it is necessary the development of more robust post-processing strategies, capable of obtain approximations for the stress in $H(\operatorname{div}, \Omega, \mathbb{M})$ and that remain accurate even in the more challenging scenario of nearly-incompressible problems. In this work, we investigate the application of the global post-processing strategy proposed in [4] on nearly-incompressible elasticity problems that, as we shall see, satisfies those requirements. To apply this post-processing strategy, we first need an accurate, previously computed, approximation for the displacement field. Here, to fix the ideas, we choose a classical mixed finite element method involving the displacement field and an auxiliary "pressure field" that is well known in the literature for its good behavior on problems near the incompressibility limit. We remark, however, that other displacement approximations methods could be used.

The paper is organized as follows. In Section 2 we present a mixed finite element method for the elasticity problem (1) that is robust in the incompressibility limit. The global post-processing strategy for the stress recovery, as well as its convergence results, are discussed in Section 3. Section 4 contains numerical experiments testing the accuracy of the proposed methods in nearly-incompressible problems. Finally, the conclusions are presented in Section 5.

## 2 Displacement approximation

Before discussing the post-processing strategy for the stress approximation studied in this work, we first need to obtain an accurate approximation for the displacement on nearly-incompressible problems. There are many possible finite element methods in the literature to achieve this goal, such as mixed methods [3, 5], discontinuous Galerkin methods [1, 6] or reduced integration methods [2, 7]. In this work we opted for one of the most classical approaches, based on a mixed variational formulation for problem (1).

Such formulation is only defined for isotropic problems, and its variables are the displacement field $\boldsymbol{u} \in H_0^1(\Omega, \mathbb{R}^2)$ and an auxiliary variable $p \in L^2(\Omega, \mathbb{R})$ defined as

$$p = \lambda \operatorname{div}\boldsymbol{u}. \tag{6}$$

From (6), it follows that the constitutive equation (4a) can be rewritten as

$$\boldsymbol{\sigma} = 2\mu\varepsilon(\boldsymbol{u}) + p\boldsymbol{I}, \tag{7}$$

where $\boldsymbol{I}$ is the identity tensor. Equation (7) and a weak form of (6) originate a mixed formulation for problem (1), given by: Find the pair $(\boldsymbol{u}, p) \in H_0^1(\Omega, \mathbb{R}^2) \times L^2(\Omega, \mathbb{R})$ such that, for all $(\boldsymbol{v}, q) \in H_0^1(\Omega, \mathbb{R}^2) \times L^2(\Omega, \mathbb{R})$

$$\int_\Omega 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{dx} + \int_\Omega p \operatorname{div}\boldsymbol{v}\,\mathrm{dx} = -\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{dx}, \tag{8a}$$

$$\int_\Omega q \operatorname{div}\boldsymbol{u}\,\mathrm{dx} - \int_\Omega \frac{1}{\lambda} pq\,\mathrm{dx} = 0. \tag{8b}$$

A mixed finite element method based on formulation (8) can be obtained by adopting suitable finite-dimensional subspaces $\boldsymbol{V}_h \times Q_h$, where $\boldsymbol{V}_h \subset H^1_0(\Omega, \mathbb{R}^2)$ and $Q_h \subset L^2(\Omega, \mathbb{R})$. This originates a discrete problem which, according to the framework presented in [2], will have a unique solution $(\boldsymbol{u}_h, p_h)$ once the spaces $\boldsymbol{V}_h$ and $Q_h$ satisfies a discrete inf-sup condition.

For the limit case $\lambda \to \infty$, which corresponds to the incompressible problem, formulation (8) has the same structure as the mixed velocity-pressure variational formulation for the Stokes problem. This motivates the informal name "pressure" for the variable $p$ since that is its physical interpretation in the Stokes context. Furthermore, the existence and uniqueness of solution for the infinite-dimensional problem (8) can be proven using the same techniques used for the Stokes formulation, and every compatible pair of finite-dimensional subspaces for Stokes will also be compatible for the elasticity mixed method presented above, as discussed in [2]. From the possible families of compatible pairs $\boldsymbol{V}_h \times Q_h$, the one used in this work is the family $\mathcal{Q}_r(\Omega, \mathbb{R}^2) \times P_{r-1}(\Omega, \mathbb{R})$ on quadrilaterals, for $r \geq 2$, which construction will be discussed next.

Let $\mathcal{T}_\mathrm{h}$ be a regular discretization of $\Omega$ into convex quadrilaterals as defined in [8, 9], $\hat{K} = [-1, 1] \times [-1, 1]$ the reference element, and $F_K$ the isomorphisms that map $\hat{K}$ onto each element $K \in \mathcal{T}_\mathrm{h}$. Denote by $P_r(K)$ the space of polynomials on $K$ of total degree less than or equal to $r$, by $P_{r,s}(\hat{K})$ the space of polynomials on the reference element $\hat{K}$ of degree at most $r$ on the first coordinate and at most $s$ on the second one, and set $Q_r(\hat{K}, \mathbb{R}^2) = P_{r,r}(\hat{K}) \times P_{r,r}(\hat{K})$. Then, the space $\mathcal{P}_r(\Omega, \mathbb{R})$ is defined as

$$\mathcal{P}_r(\Omega, \mathbb{R}) = \{q \in L^2(\Omega) \,:\, q|_K \in P_r(K), \forall K \in \mathcal{T}_\mathrm{h}\},$$

while the space $\mathcal{Q}_r(\Omega, \mathbb{R}^2)$ is given by

$$\mathcal{Q}_r(\Omega, \mathbb{R}^2) = \{\boldsymbol{v} \in H^1_0(\Omega, \mathbb{R}^2) \,:\, \boldsymbol{v}|_K \in \mathcal{Q}_r(K, \mathbb{R}^2), \forall K \in \mathcal{T}_\mathrm{h}\},$$

where

$$\mathcal{Q}_r(K, \mathbb{R}^2) = \{\boldsymbol{v} \in H^1(K, \mathbb{R}^2) \,:\, \boldsymbol{v} = \hat{\boldsymbol{v}} \circ F_K^{-1}, \hat{\boldsymbol{v}} \in Q_r(\hat{K}, \mathbb{R}^2)\}.$$

It follows, from the analysis presented in [2, 9], that the spaces $\mathcal{Q}_r(\Omega, \mathbb{R}^2) \times \mathcal{P}_{r-1}(\Omega, \mathbb{R})$, with $r \geq 2$, guarantee that the discrete problem associated to formulation (8) has a unique solution $(\boldsymbol{u}_h, p_h)$. Additionally, if the exact solution $(\boldsymbol{u}, p)$ is regular enough, the approximated solution satisfies the following error estimates

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} \leq Ch^{r+1}(\|\boldsymbol{u}\|_{r+1,\Omega} + \|p\|_{r,\Omega}), \tag{9a}$$

$$\|\boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,\Omega} \leq Ch^{r}(\|\boldsymbol{u}\|_{r+1,\Omega} + \|p\|_{r,\Omega}), \tag{9b}$$

$$\|p - p_h\|_{0,\Omega} \leq Ch^{r}(\|\boldsymbol{u}\|_{r+1,\Omega} + \|p\|_{r,\Omega}), \tag{9c}$$

where $h$ is the mesh parameter, defined as the maximum of the diameters of the elements $K \in \mathcal{T}_\mathrm{h}$.

In addition to the good performance on nearly-incompressible problems and the optimal convergence results (9), another advantage of this mixed method is its computational cost. It is possible to show that the degrees of freedom associated with the variable $p_h$ can be condensed so that the global system is only solved in terms of $\boldsymbol{u}_h$. Therefore, the overall computational cost of the method is similar to that of the continuous Galerkin method.

## 3 Global stress post-processing

In this section, providing a previously known approximation for the displacement (or more precisely, for $\boldsymbol{\varepsilon}(\boldsymbol{u})$), we present a global post-processing strategy to approximate the stress tensor. This strategy consists of solving a new global problem, constructed by combining a weak form of the constitutive equation (4a) with least-square residuals of the balance equation (4b). Although this strategy was originally described in [4] for the more general context of elliptic problems, in this work we will focus only on its application to the linear elasticity problem, as was done recently in [10].

Let $\mathcal{T}_\mathrm{h}$ be the mesh in which an approximation $\boldsymbol{u}_h$ for the displacement was previously computed. Provided $\boldsymbol{Z}_h \subset H(\mathrm{div}, \Omega, \mathbb{M})$ a finite-dimensional subspace, the new global problem is defined by: Find $\boldsymbol{\sigma}_h \in \boldsymbol{Z}_h$ such that

$$b_\alpha(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}_h) : \boldsymbol{\tau}_h \,\mathrm{dx} + (\delta h)^\alpha \int_\Omega \boldsymbol{f} \cdot \mathrm{div}\, \boldsymbol{\tau}_h \,\mathrm{dx}, \quad \forall \boldsymbol{\tau}_h \in \boldsymbol{Z}_h, \tag{10}$$

where $b_\alpha : H(\mathrm{div}, \Omega, \mathbb{M}) \times H(\mathrm{div}, \Omega, \mathbb{M}) \to \mathbb{R}$ is a bilinear form given by

$$b_\alpha(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) = \int_\Omega \boldsymbol{A}\boldsymbol{\sigma}_h : \boldsymbol{\tau}_h \,\mathrm{dx} + (\delta h)^\alpha \int_\Omega \mathrm{div}\, \boldsymbol{\sigma}_h \cdot \mathrm{div}\, \boldsymbol{\tau}_h \,\mathrm{dx}, \tag{11}$$

$h$ is the mesh parameter associated with $\mathcal{T}_\mathrm{h}$, and $\delta > 0$ and $0 \leq \alpha \leq 2$ are parameters to be chosen.

Among the several possible choices for the subspaces $\boldsymbol{Z}_h$, in this work, we will only consider subspaces composed of tensors whose rows are vectors in the well-known Raviart-Thomas spaces [11]. To construct these spaces we first need to introduce the Piola transform. For vector functions $\hat{\boldsymbol{v}} : \hat{K} \to \mathbb{R}^2$, the Piola transform of $\hat{\boldsymbol{v}}$, denoted by $\mathcal{P}_{F_K}\hat{\boldsymbol{v}} : K \to \mathbb{R}^2$, is

$$\mathcal{P}_{F_K}\hat{\boldsymbol{v}} = (JF_K)^{-1} DF_K(\hat{\boldsymbol{x}})\hat{\boldsymbol{v}}(\hat{\boldsymbol{x}}),$$

with $DF_K$ being the Jacobian matrix of $F_K$ and $JF_K$ its determinant. We shall define the Piola transform of a tensor $\hat{\boldsymbol{\tau}} : \hat{K} \to \mathbb{M}$ as the vectorial Piola transform applied in each row of the tensor, i.e,

$$(\mathcal{P}_{F_K}\hat{\boldsymbol{\tau}})_i = (JF_K)^{-1} DF_K(\hat{\boldsymbol{x}})\hat{\boldsymbol{\tau}}_i(\hat{\boldsymbol{x}}),$$

where the index $i$ denotes the i-th row of the tensors.

For regular meshes made of convex quadrilaterals, the "tensorial Raviart-Thomas space" of index $l \geq 0$ is defined as

$$\boldsymbol{Z}_h = \mathcal{RT}_l(\Omega, \mathbb{M}) = \{\boldsymbol{\tau} \in H(\mathrm{div}, \Omega, \mathbb{M}) \ : \ \boldsymbol{\tau}|_K \in \mathcal{P}_{F_K} RT(\hat{K}, \mathbb{M}), \quad \forall K \in \mathcal{T}_{\mathrm{h}}\}. \tag{12}$$

where $RT(\hat{K}, \mathbb{M})$ is the space of tensors whose rows are vectors in $P_{l+1,l}(\hat{K}) \times P_{l,l+1}(\hat{K})$.

By using the strategy (10) with the Raviard-Thomas based spaces (12) we not only guarantee that $\boldsymbol{\sigma}_h$ belongs to $H(\mathrm{div}, \Omega, \mathbb{M})$, but also achieve good approximation results in the norms $L^2$ and $H(\mathrm{div})$, as we will briefly discuss in the next section.

## 3.1 Convergence results

Here we will present the main convergence results associated with the post-processing strategy (10). The proof of these results are similar to the ones presented in [4] with minor modifications, and therefore will be omitted for the sake of brevity. We begin by defining a norm $\|\cdot\|_\alpha$ over $H(\mathrm{div}, \Omega, \mathbb{M})$ induced by the bilinear form $b_\alpha(\cdot, \cdot)$

$$\|\boldsymbol{\tau}\|_\alpha = (b_\alpha(\boldsymbol{\tau}, \boldsymbol{\tau}))^{1/2}, \tag{13}$$

and noticing the the existence of a constant $C$ such that

$$\|\boldsymbol{\tau}\|_\alpha \geq \ C\|\boldsymbol{\tau}\|_0, \quad \text{and} \quad \|\boldsymbol{\tau}\|_\alpha \geq \ (\delta h)^{\alpha/2}\|\,\mathrm{div}\,\boldsymbol{\tau}\|_0. \tag{14}$$

It then follows that, for every subspace $\boldsymbol{Z}_h$ of $H(\mathrm{div}, \Omega, \mathbb{M})$ and every approximation $\boldsymbol{u}_h$ for the displacement, there is a constant $C$ (independent of the mesh parameter $h$ and the Lamé coeficient $\lambda$), such that the approximated stress obtained through (10) satisfies

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_\alpha \leq 2\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_\alpha + C\|\varepsilon(\boldsymbol{u} - \boldsymbol{u}_h)\|_0, \quad \forall \boldsymbol{\tau}_h \in \boldsymbol{Z}_h. \tag{15}$$

Now we will particularize the more general result (15) for the case where $\boldsymbol{u}_h$ was obtained through the mixed method (8) and $\boldsymbol{Z}_h = \mathcal{RT}_l(\Omega, \mathbb{M})$. For doing so, we will also assume that the exact solutions for the displacement and stress fields are regular enough and the discretization $\mathcal{T}_{\mathrm{h}}$ of $\Omega$ is regular and composed only by parallelogram elements. Under those assumptions, using the result (15), the estimate (9b), approximation properties of the spaces $\mathcal{RT}_l(\Omega, \mathbb{M})$ on parallelogram meshes, and the relations (14), it follows that

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \leq C(h^r(\|\boldsymbol{u}\|_{r+1,\Omega} + \|p\|_{r,\Omega}) + h^{l+1}(|\boldsymbol{\sigma}|_{l+1,\Omega} + |\,\mathrm{div}\,\boldsymbol{\sigma}|_{l+1,\Omega})), \tag{16a}$$

$$\|\,\mathrm{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega} \leq C(h^{r-\alpha/2}(\|\boldsymbol{u}\|_{r+1,\Omega} + \|p\|_{r,\Omega}) + h^{l+1-\alpha/2}(|\boldsymbol{\sigma}|_{l+1,\Omega} + |\,\mathrm{div}\,\boldsymbol{\sigma}|_{l+1,\Omega})). \tag{16b}$$

**Remark 3.1.** *For the case of meshes composed by general convex quadrilaterals (such as trapezoids), the approximation capability of the spaces $\mathcal{RT}_l(\Omega, \mathbb{M})$ is sub-optimal on the norms $\|\cdot\|_\alpha$ and $\|\cdot\|_{H(\mathrm{div})}$, as can be shown from the results of [8]. The implications of this result on the post-processing strategy will be discussed in a future work. In the present work, we will retain our analysis to the simpler case of parallelogram meshes.*

Estimates (16a) and (16b) indicates that, provide that $\boldsymbol{u}_h$ satisfies (9b) and $\mathcal{T}_{\mathrm{h}}$ is a regular mesh made of parallelograms, the best convergence rates for the stress approximation are achieved setting $l = r - 1$ and $\alpha = 0$, for which $\mathcal{O}(h^r)$ rates are obtained for both the $L^2$ and the $H(\mathrm{div})$ norms. This will, in fact, be our choice for the numerical experiments done in the next section, where we check those convergence results through a test problem.

# 4 Numerical experiments

In this section, we perform some numerical experiments aiming to check the theoretical estimates (16a) and (16b) and evaluate the performance of the presented combination of methods (mixed method displacement-pressure for the displacement approximation and global post-processing technique for the stress approximation) on nearly-incompressible problems. To do so, we take the model problem (1) and set $\Omega = (-1,1)\times(-1,1)$, $\mu = 1$ and

$$\boldsymbol{f}(x,y) = \pi^2 \begin{bmatrix} -4\sin(2\pi y)(2\cos(2\pi x)-1) + \cos(\pi(x+y)) - \frac{2}{1+\lambda}\sin(\pi x)\sin(\pi y) \\ -4\sin(2\pi x)(1-2\cos(2\pi y)) + \cos(\pi(x+y)) - \frac{2}{1+\lambda}\sin(\pi x)\sin(\pi y) \end{bmatrix}.$$

Such test problem was proposed in [12] and its exact solution for the displacement is given by

$$\boldsymbol{u}(x,y) = \begin{bmatrix} \sin(2\pi y)(\cos(2\pi x)-1) + \frac{1}{1+\lambda}\sin(\pi x)\sin(\pi y) \\ \sin(2\pi x)(1-\cos(2\pi y)) + \frac{1}{1+\lambda}\sin(\pi x)\sin(\pi y) \end{bmatrix}.$$

It is easy to verify that although the exact solutions for the displacement and stress depend on $\lambda$, both are bounded as $\lambda$ tends to infinity. The values for $\lambda$ used in our experiments are obtained from the Poisson's constant $\nu$ according to

$$\lambda = \frac{\nu}{(1+\nu)(1-2\nu)}, \tag{17}$$

and three values are considered: $\nu = 0.3$, $0.4999$ and $0.4999999$. The first value for $\nu$ leads to a compressible problem, while the choices $0.4999$ and $0.4999999$ are progressive closer to the incompressibility limit.

For the displacement approximation, we use the mixed method (8) with the compatible approximation spaces $\mathcal{Q}_2(\Omega,\mathbb{R}^2)\times\mathcal{P}_1(\Omega,\mathbb{R})$. The approximated solutions $\boldsymbol{u}_h$ are obtained for a sequence of meshes $\mathcal{T}_h$ of $\Omega$ made of $n\times n$ square elements, with $n = 16$, 32, 64 and 128. Table 1 shows the errors obtained in the approximation of $\boldsymbol{u}$, $\nabla\boldsymbol{u}$, $\varepsilon(\boldsymbol{u})$, and $p$ in the $L^2$ norm and their respective convergence rates. As we can see, the displacement approximation remains accurate even in the nearly-incompressible cases, with absolute errors $\|\boldsymbol{u}-\boldsymbol{u}_h\|_0$ and $\|\varepsilon(\boldsymbol{u}-\boldsymbol{u}_h)\|_0$ of the same magnitude as the ones obtained in the case $\nu = 0.3$. We also notice that the convergence rates found numerically agree in totality with the theoretical estimates (9).

Table 1. $L^2$ errors, and their respective convergence rates, in the displacement approximation using the mixed method with approximation spaces $\mathcal{Q}_2(\Omega,\mathbb{R}^2)\times P_1(\Omega,\mathbb{R})$.

| | $\|\boldsymbol{u}-\boldsymbol{u}_h\|_0$ | | $\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_0$ | | $\|\varepsilon(\boldsymbol{u}-\boldsymbol{u}_h)\|_0$ | | $\|p-p_h\|_0$ | |
|---|---|---|---|---|---|---|---|---|
| n | err. | rate | err. | rate | err. | rate | err. | rate |
| Case $\nu = 0.3$ ($\lambda \approx 5.77\cdot10^{-1}$) | | | | | | | | |
| 16 | 7.741e-03 | 2.92 | 4.077e-01 | 1.98 | 3.211e-01 | 1.96 | 2.493e-02 | 2.00 |
| 32 | 9.817e-04 | 2.98 | 1.022e-01 | 2.00 | 8.074e-02 | 1.99 | 6.196e-03 | 2.01 |
| 64 | 1.232e-04 | 2.99 | 2.557e-02 | 2.00 | 2.021e-02 | 2.00 | 1.546e-03 | 2.00 |
| 128 | 1.541e-05 | 3.00 | 6.393e-03 | 2.00 | 5.055e-03 | 2.00 | 3.863e-04 | 2.00 |
| Case $\nu = 0.4999$ ($\lambda \approx 1.67\cdot10^{3}$) | | | | | | | | |
| 16 | 7.724e-03 | 2.92 | 4.074e-01 | 1.98 | 3.207e-01 | 1.97 | 7.038e-02 | 2.08 |
| 32 | 9.800e-04 | 2.98 | 1.021e-01 | 2.00 | 8.061e-02 | 1.99 | 1.708e-02 | 2.04 |
| 64 | 1.230e-04 | 2.99 | 2.553e-02 | 2.00 | 2.018e-02 | 2.00 | 4.233e-03 | 2.01 |
| 128 | 1.539e-05 | 3.00 | 6.383e-03 | 2.00 | 5.046e-03 | 2.00 | 1.056e-03 | 2.00 |
| Case $\nu = 0.4999999$ ($\lambda \approx 1.67\cdot10^{6}$) | | | | | | | | |
| 16 | 7.724e-03 | 2.92 | 4.074e-01 | 1.98 | 3.207e-01 | 1.97 | 7.042e-02 | 2.08 |
| 32 | 9.800e-04 | 2.98 | 1.021e-01 | 2.00 | 8.061e-02 | 1.99 | 1.709e-02 | 2.04 |
| 64 | 1.230e-04 | 2.99 | 2.553e-02 | 2.00 | 2.018e-02 | 2.00 | 4.235e-03 | 2.01 |
| 128 | 1.538e-05 | 3.00 | 6.383e-03 | 2.00 | 5.046e-03 | 2.00 | 1.056e-03 | 2.00 |

Next, we use the results obtained by the mixed method for $\varepsilon(\boldsymbol{u}_h)$ as an input for the global post-processing strategy (10). From the analysis presented in Section 3.1, by using the space $\mathcal{RT}_1(\Omega, \mathbb{M})$ and setting $\alpha = 0$, we can achieve optimal convergence errors $\mathcal{O}(h^2)$ in the stress approximation on both $L^2$ and $H(\text{div})$ norms. This result is verified numerically in the second and third columns of Table 2. From that same table, we remark that the approximations for the stress using strategy (10) remain accurate even in the nearly-incompressible cases.

We also present, in the first column of Table 2, approximation results for the stress obtained directly from the $\varepsilon(\boldsymbol{u}_h)$ and the local application of the constitutive equation, as defined by the simpler approach (5). Even though the approximations $\varepsilon(\boldsymbol{u}_h)$ were precise, the stress approximated by (5) deteriorates as $\lambda$ tends to infinity, reaching prohibitive absolute errors in the cases $\nu = 0.4999$ and $\nu = 0.4999999$. This result reinforces the necessity of a suitable strategy for the stress approximation on nearly-incompressible problems.

Table 2. $L^2$ errors, and their respective convergence rates, in the stress approximation using the post-processing strategy with the approximation space $\mathcal{RT}_1(\Omega, \mathbb{M})$.

| | $\|\boldsymbol{\sigma} - \boldsymbol{C}\varepsilon(\boldsymbol{u}_h)\|_0$ | | $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0$ | | $\|\operatorname{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega}$ | |
|---|---|---|---|---|---|---|
| n | err. | rate | err. | rate | | |
| Case $\nu = 0.3$ ($\lambda \approx 5.77 \cdot 10^{-1}$) | | | | | | |
| 16 | 7.289e-01 | 1.95 | 5.031e-01 | 2.00 | 4.032e+00 | 1.96 |
| 32 | 1.838e-01 | 1.99 | 1.257e-01 | 2.00 | 1.015e+00 | 1.99 |
| 64 | 4.605e-02 | 2.00 | 3.142e-02 | 2.00 | 2.541e-01 | 2.00 |
| 128 | 1.152e-02 | 2.00 | 7.855e-03 | 2.00 | 6.356e-02 | 2.00 |
| Case $\nu = 0.4999$ ($\lambda \approx 1.67 \cdot 10^{3}$) | | | | | | |
| 16 | 4.698e+02 | 1.89 | 5.013e-01 | 1.99 | 4.026e+00 | 1.96 |
| 32 | 1.197e+02 | 1.97 | 1.254e-01 | 2.00 | 1.013e+00 | 1.99 |
| 64 | 3.006e+01 | 1.99 | 3.135e-02 | 2.00 | 2.537e-01 | 2.00 |
| 128 | 7.524e+00 | 2.00 | 7.838e-03 | 2.00 | 6.346e-02 | 2.00 |
| Case $\nu = 0.4999999$ ($\lambda \approx 1.67 \cdot 10^{6}$) | | | | | | |
| 16 | 4.696e+05 | 1.89 | 5.013e-01 | 1.99 | 4.026e+00 | 1.96 |
| 32 | 1.196e+05 | 1.97 | 1.254e-01 | 2.00 | 1.013e+00 | 1.99 |
| 64 | 3.005e+04 | 1.99 | 3.135e-02 | 2.00 | 2.537e-01 | 2.00 |
| 128 | 7.520e+03 | 2.00 | 7.838e-03 | 2.00 | 6.346e-02 | 2.00 |

## 5 Conclusions

In this work, we studied the combination of a mixed displacement-pressure method with a global post-processing technique to obtain approximated solutions for the displacement and stress fields in the context of linear elasticity problems. The convergence results presented and the numerical experiments performed suggest that the combination is suitable for those problems, even in the more challenging scenario of nearly-incompressible materials. Not only optimal convergence rates were achieved, but also small absolute errors were obtained, showing the combination's accuracy.

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## References

[1] P. Hansbo and M. G. Larson. Discontinuous galerkin methods for incompressible and nearly incompressible elasticity by nitsche's method. *Computer methods in applied mechanics and engineering*, vol. 191, n. 17-18, pp. 1895–1908, 2002.

[2] D. Boffi, F. Brezzi, M. Fortin, and others. *Mixed finite element methods and applications*, volume 44. Springer, 2013.

[3] D. N. Arnold. Mixed finite element methods for elliptic problems. *Computer methods in applied mechanics and engineering*, vol. 82, n. 1-3, pp. 281–300, 1990.

[4] A. F. Loula, F. A. Rochinha, and M. A. Murad. Higher-order gradient post-processings for second-order elliptic problems. *Computer Methods in Applied Mechanics and Engineering*, vol. 128, n. 3-4, pp. 361–381, 1995.

[5] T. O. Quinelato, A. F. Loula, M. R. Correa, and T. Arbogast. Full h (div)-approximation of linear elasticity on quadrilateral meshes based on abf finite elements. *Computer Methods in Applied Mechanics and Engineering*, vol. 347, pp. 120–142, 2019.

[6] B. Cockburn, D. Schötzau, and J. Wang. Discontinuous galerkin methods for incompressible elastic materials. *Computer methods in applied mechanics and engineering*, vol. 195, n. 25-28, pp. 3184–3204, 2006.

[7] D. S. Malkus and T. J. Hughes. Mixed finite element methods—reduced and selective integration techniques: a unification of concepts. *Computer Methods in Applied Mechanics and Engineering*, vol. 15, n. 1, pp. 63–81, 1978.

[8] D. N. Arnold, D. Boffi, and R. S. Falk. Quadrilateral h (div) finite elements. *SIAM Journal on Numerical Analysis*, vol. 42, n. 6, pp. 2429–2451, 2005.

[9] V. Girault and P.-A. Raviart. *Finite element methods for Navier-Stokes equations: theory and algorithms*, volume 5. Springer Science & Business Media, 2012.

[10] G. Taraschi, A. S. Pinto, C. O. Faria, and M. R. Correa. On the accuracy of finite element approximations of elliptic problems with heterogeneous coefficients. *Proceedings of the Ibero-Latin-American Congress on Computational Methods in Engineering*, vol. 1, 2021.

[11] P.-A. Raviart and J.-M. Thomas. A mixed finite element method for 2-nd order elliptic problems. In *Mathematical aspects of finite element methods*, pp. 292–315. Springer, 1977.

[12] S. C. Brenner. A nonconforming mixed multigrid method for the pure displacement problem in planar linear elasticity. *SIAM Journal on Numerical Analysis*, vol. 30, n. 1, pp. 116–135, 1993.