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Effective elastic properties of fractured materials by means of a homogenization approach

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Abstract. A main characteristic of many engineering materials is the presence of natural fractures at different scales. The effective mechanical behavior of these materials is strongly affected by that of the fractures, which can be viewed as discontinuities able to transfer stresses. The contribution of the present work relies upon micromechanics for assessing the effective stiffness through homogenization upscaling of elastic materials with embedded microfractures. In the context of Eshelby equivalent inclusion theory, the approach makes use of the Mori-Tanaka scheme to formulate estimates for the homogenized elastic moduli. For this purpose, the fractures are geometrically modeled as oblate spheroids endowed with appropriate elastic properties. Particular emphasis is dedicated to addressing the configurations of a single family or two families of parallel fractures, as well as the configuration of randomly oriented fractures.

Keywords: Microfracture, Micromechanics, Homogenization, Effective elasticity

1 Introduction

Most of engineering materials and especially geomaterials such as rock, concrete, or asphalt pavements exhibit discontinuity surfaces at different scales with various sizes and orientations. Usually, these discontinuities are associated with fractures and correspond to a region of small thickness, along which the mechanical and physical properties of the material are degraded. Their presence constitutes a fundamental weak component for deformability, stability, and safety of many civil engineering structures, reducing stiffness, shear strength, and ductility, in addition to providing preferential channels for fluid flow. Unlike cracks, fractures are discontinuities that are able to transfer stresses, and can therefore be regarded from a mechanical viewpoint as interfaces endowed with a specific behavior under normal and shear loading.

This paper will focus on the particular case of micro-fractures, i.e., discontinuities with small extension when compared to the size of the representative elementary volume of the material. The homogenization-based approaches provide an appropriate framework for constitutive modeling whenever the network if discontinuities present in the medium is sufficiently dense. In this perspective, strength, deformation, and permeability coupling of cracked and fractured materials have been widely investigated in literature during past decade. Representative works include references [1–10], to cite a few.

The main purpose of the present contribution is to extend the formulation proposed by Maghous et al. [7], related to the particular case of a single family of short fractures distributed parallel in the matrix, to a configuration of two families of short fractures. Furthermore, particular emphasis is given to the randomly oriented microfractures (isotropic case) already demonstrated in Maghous et al. [8] and Aguiar and Maghous [10].

2 Micromechanics

The description of the mechanical behavior of heterogeneous media is a complex task from the mathematical viewpoint due to the morphological complexity of these materials. In this sense, the homogenization theory has proven to be an efficient tool, because it allows transforming the heterogeneous medium into an equivalent homogeneous medium and consequently simplifies the mathematical treatment of the problem. Since microfractures are randomly distributed in the medium, to apply the homogenization theory to microfractured materials it is necessary to considers a volume containing a sufficiently large number of microfractures so that the volume statistically represents the average behavior of the material anywhere in the structure. This volume is called representative elementary volume (REV) and must meet the scale separation condition: ($d \ll l \ll L$). The typical length scale l of the REV should be small enough as compared to the characteristic dimension L of the whole structure, so as to enable the use of the differential tools of continuum mechanics. In addition, l should also be large enough as compared to the characteristic length d of the heterogeneities to ensure statistical representativeness [4, 9].

2.1 Hill's lemma for the fractured media

We will use the notation Ω to represent all the domain of the REV and ω to characterize all the volume of the microfractures. At the scale of the REV (microscopic scale), each fracture is modeled as an interface ω_i, in which the orientation is defined by a normal unit vector $\underline{n}_i$, as represented by the Fig. 1.

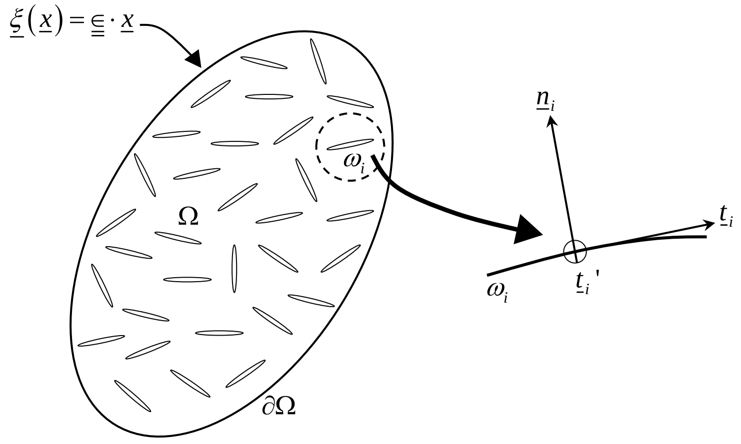

Figure 1. Representative elementary volume and loading mode (Adapted from Aguiar and Maghous [10])

The solid matrix fills the domain $\Omega \backslash \omega$, where symbol "\" stands for the set difference. Note that strains and stresses within the heterogeneous medium are defined on the solid matrix domain $\Omega \backslash \omega$ only, and not on the whole REV. Throughout the paper, symbol $\langle \cdot \rangle$ denotes the volume average over the solid matrix, as indicated by [8, 10]:

$$\langle \cdot \rangle = \frac{1}{|\Omega|} \int_{\Omega \backslash \omega} \cdot \, dV. \tag{1}$$

The loading applied to the REV (see Fig. 1) is defined by homogeneous strain type boundary conditions on the boundary $\partial\Omega$:

$$\underline{\xi}\,(\underline{x}) = \underline{\underline{\in}} \cdot \underline{x} \quad \forall \underline{x} \in \partial\Omega \tag{2}$$

where $\underline{\xi}$ represents the displacement field, $\underline{\underline{\in}}$ is the macroscopic strain, and $\underline{x}$ is the position vector.

Hill's lemma extended to the particular situation of fractured medium, established by Maghous et al. [7], for any statically admissible stress field $\underline{\underline{\sigma}}$ and any kinematically admissible displacement field $\underline{\xi}$, takes the form:

$$\langle \underline{\underline{\sigma}} \rangle : \underline{\underline{\in}} = \langle \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}} \rangle_{\Omega \backslash \omega} + \frac{1}{|\Omega|} \int_{\omega} \underline{T} \cdot \left[\underline{\xi}\right] dS \tag{3}$$

in which, the term $\underline{T}$ is the surface forces acting on the faces of each fracture and $\left[\underline{\xi}\right]$ is the displacement jump observed at each discontinuity. Since Hill's Lemma is valid for any stress and strain fields, not necessarily correlated, taking a symmetric and uniform tensor $\underline{\underline{\sigma}}$, the macroscopic strain can be written as:

$$\underline{\underline{\in}} = \langle \underline{\underline{\varepsilon}} \rangle + \frac{1}{|\Omega|} \int_{\omega} \left[\underline{\xi}\right] \overset{s}{\otimes} \underline{n} \, dS \tag{4}$$

where symbol $\overset{s}{\otimes}$ refers to the symmetric part of dyadic product: $\left(\underline{u} \overset{s}{\otimes} \underline{v}\right)_{ij} = \left(u_i v_j + v_i u_j\right)/2$ and $\underline{n} = \underline{n}_i$ along ω_i. Therefore, eq. (4) shows that the macroscopic strain contains two contributions, one from the solid matrix and the other one from the discontinuities.

2.2 Elastic Homogenization Framework

This section describes the homogenized elastic behavior of a medium with a short fracture distribution. In the context of linear elasticity, the stress $\underline{\underline{\sigma}}$ and strain $\underline{\underline{\varepsilon}}$ fields in the matrix are related by means of the fourth-order stiffness tensor $\mathbb{C}^s$. While the displacement jump $\left[\underline{\xi}\right]$ is related to force vector $\underline{T}$ through elastic stiffness $\underline{\underline{k}} = \underline{\underline{k}}^i$ as a function of the local coordinate system $(\underline{t}_i, \underline{t}'_i, \underline{n}_i)$ of each fracture ω_i, illustrated in Fig. 1 [11, 12]. Both relationships are presented in the eq. (5):

$$\begin{cases} \underline{\underline{\sigma}} = \mathbb{C}^s : \underline{\underline{\varepsilon}} & \text{in} \quad \Omega \backslash \omega \qquad \text{(a)} \\ \underline{T} = \underline{\underline{\sigma}} \cdot \underline{n} = \underline{\underline{k}} \cdot \left[\underline{\xi}\right] & \text{along} \quad \omega = \bigcup_i \omega_i \quad \text{(b)} \end{cases} \tag{5}$$

For the sake of clarity, except when necessary, the subscript (i) referring to fracture ω_i will be omitted in subsequent developments. In particular, $(\underline{t}, \underline{t}', \underline{n})$ will generically denote the orthonormal frame related to any fracture of the set $\omega = \bigcup_i \omega_i$. Therefore, the tensor $\underline{\underline{k}}$ will be expressed in the local coordinate system as:

$$\underline{\underline{k}} = k_n \ \underline{n} \otimes \underline{n} + k_t \left(\underline{t} \otimes \underline{t} + \underline{t}' \otimes \underline{t}'\right) \tag{6}$$

where k_n and k_t respectively represent the normal and tangential stiffness components of the fracture, expressed in (Pa/m). These quantities are usually obtained from laboratory tests performed on specimens containing a single fracture. Further details related to the physical interpretation and identification procedures of these parameters are presented in Bandis et al. [11] or Goodman [12]. Note that the particular case of discontinuities that do not transmit stresses (cracks) can be included in the above formulation by considering a null value for the corresponding stiffness ($k_n = 0$ and $k_t = 0$).

The loading condition (eq. (2)) associated with the state equations (eq. (5)) define the pair $\left(\underline{\underline{\sigma}}, \underline{\xi}\right)$ as the solution of the elastic concentration problem. Considering that in the homogenized medium the macroscopic strain $\underline{\underline{\in}}$ is constant on the boundary of the `REV`, it can be shown (see Zaoui [2]) that the local strains $\underline{\underline{\varepsilon}}\left(\underline{x}\right)$ are linearly related to the equivalent strains $\underline{\underline{\in}}$ according to:

$$\underline{\underline{\varepsilon}}\left(\underline{x}\right) = \mathbb{A}\left(\underline{x}\right) : \underline{\underline{\in}} \tag{7}$$

in which $\mathbb{A}$ characterizes the fourth-order strain concentration tensor. In contrast to the continuous framework in which relationship $\langle \mathbb{A} \rangle = \mathbb{I}$ is valid, the analysis involving discontinuities takes the average of the concentration tensor over the `REV` to a non-unitary value $\langle \mathbb{A} \rangle \neq \mathbb{I}$ (see Maghous et al. [7]).

The classical reasoning in linear elastic homogenization describes the macroscopic elastic behavior law of a fractured medium by means of relationship relative stresses $\underline{\underline{\Sigma}}$ and strains $\underline{\underline{\in}}$ at the macroscopic scale through the fourth-order homogenized elastic stiffness tensor $\mathbb{C}^{\text{hom}}$ [2]:

$$\underline{\underline{\Sigma}} = \mathbb{C}^{\text{hom}} : \underline{\underline{\in}} \quad \text{with} \quad \mathbb{C}^{\text{hom}} = \langle \mathbb{C}^s : \mathbb{A} \rangle. \tag{8}$$

When a composite consists of a matrix phase (s) and n phases of inclusions (i), the eq. (8), referring to $\mathbb{C}^{\text{hom}}$, takes the following expression:

$$\mathbb{C}^{\text{hom}} = \mathbb{C}^s + \sum_{i=1}^{n} f_i \left(\mathbb{C}^i - \mathbb{C}^s\right) : \langle \mathbb{A} \rangle_{\Omega_i} \tag{9}$$

in which f_i and $\mathbb{C}^i$ are respectively the volume fraction and the elastic stiffness tensor of the phase (i), while $\langle \mathbb{A} \rangle_{\Omega_i}$ represents the average of the strain concentration tensor in phase (i).

In the context of Eshelby's equivalent inclusion theory, the approach makes use of the Mori-Tanaka scheme to determine $\langle \mathbb{A} \rangle_{\Omega_i}$ and formulates the general estimate for the homogenized elastic moduli:

$$\mathbb{C}^{\text{hom}}_{\text{MT}} = \mathbb{C}^s + \sum_{i=1}^{n} f_i \left(\mathbb{C}^i - \mathbb{C}^s\right) : \left[\mathbb{I} + \mathbb{P} : \left(\mathbb{C}^i - \mathbb{C}^s\right)\right]^{-1} : \left\langle \left[\mathbb{I} + \mathbb{P} : \left(\mathbb{C}^i - \mathbb{C}^s\right)\right]^{-1} \right\rangle_{\Omega}^{-1} \tag{10}$$

where $\mathbb{P}$ defines the Hill tensor and $\mathbb{I}$ characterizes the fourth-order identity tensor.

3 Homogenized elastic properties of fractured material

In this section we will use eq. (10) to address particular cases of materials with embedded microfractures, aiming to obtain their homogenized elastic properties. From this perspective, fractures are modeled geometrically as oblate ellipsoids (spheroids) and their orientation is associated with the orthonormal frame $(\underline{t}, \underline{t}', \underline{n})$. As illustrated in Fig. 2, the shape of this spheroid is defined by its largest radius $a = a_1 = a_2$, and by its smallest radius $c = a_3$, which determines its thickness. The aspect ratio of the spheroid is given as $X = c/a$ and for them to correctly represent the fracture geometry it is necessary that $X \ll 1$.

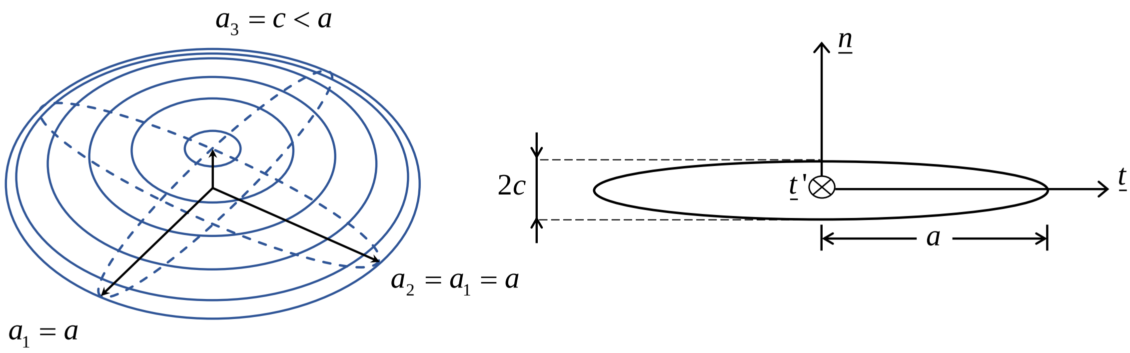

Figure 2. Fracture modeled as oblate spheroid

Considering that the solid matrix material is homogeneous and isotropic, the associated stiffness tensor takes the following form:

$$\mathbb{C}^s = 3k^s \mathbb{J} + 2\mu^s \mathbb{K} \tag{11}$$

where k^s and μ^s are respectively, the bulk and shear modulus. The fourth-order tensors $\mathbb{J}$ and $\mathbb{K}$ are defined as:

$$\mathbb{J} = \frac{1}{3} \underline{\underline{1}} \otimes \underline{\underline{1}} \quad \text{and} \quad \mathbb{K} = \mathbb{I} - \mathbb{J}. \tag{12}$$

First we consider a single fracture family, i.e., all fractures of this family have the same properties, oriented along the fixed normal unit vector $\underline{n}$, leading to a solution with parallel fractures distributed in the matrix. The volume fraction of fractures in the medium is defined by:

$$f = \frac{4}{3} \pi \epsilon X \tag{13}$$

where $\epsilon = \mathcal{N} a^3$ represents the fracture density parameter associated with the set of parallel fractures [1, 4], which can be seen as the damage parameter on the macroscopic scale and $\mathcal{N}$ is the number of fractures per unit volume.

Using the Mori-Tanaka scheme, the expression of $\mathbb{C}^{\text{hom}}$ for the fractured medium is given by eq. (14) such as [7, 8]:

$$\mathbb{C}^{\text{hom}} = \lim_{X\to 0} \left[\left(\mathbb{C}^s + f\,\mathbb{C}^f : \left(\mathbb{I} + \mathbb{P} : \left(\mathbb{C}^f - \mathbb{C}^s\right)\right)^{-1} \right) : \left(\mathbb{I} + f\left(\mathbb{I} + \mathbb{P} : \left(\mathbb{C}^f - \mathbb{C}^s\right)\right)^{-1} \right)^{-1} \right] \tag{14}$$

in which $\mathbb{P} = \mathbb{P}\left(X, \underline{n}\right)$ is the Hill tensor relative to the parallel fracture family, that depends on the aspect factor X and the orientation $\underline{n}$ of oblate spheroids. The components of the Hill tensor can be found in Handbooks (see Eshelby [13], Mura [14] or Nemat-Nasser and Hori [15]). At eq. (14), the tensor $\mathbb{C}^f$ relates to fracture stiffness $\underline{\underline{k}}$ by means of [7]:

$$\mathbb{C}^f = 3Xa\left(k_n - \frac{4}{3}k_t\right)\mathbb{J} + 2Xak_t\mathbb{K}. \tag{15}$$

The estimate derived from the Mori-Tanaka scheme (eq. (14)), considering that the frame $(\underline{t}, \underline{t}', \underline{n})$ coincides with $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ is defined by the following non-zero components of the equivalent elastic stiffness tensor $\mathbb{C}^{\text{hom}}$:

$$\begin{aligned}
C^{\text{hom}}_{1111} = C^{\text{hom}}_{2222} &= (3k^s + 4\mu^s)\,\frac{\kappa_2 + \pi\left(1 + \frac{16}{3}\epsilon\right)\kappa_1\left(1-\kappa_1\right)}{3\kappa_2 + 3\pi\kappa_1\left(1-\kappa_1\right) + 4\pi\epsilon}\\
C^{\text{hom}}_{3333} &= (3k^s + 4\mu^s)\,\frac{\kappa_2 + \pi\kappa_1\left(1-\kappa_1\right)}{3\kappa_2 + 3\pi\kappa_1\left(1-\kappa_1\right) + 4\pi\epsilon}\\
C^{\text{hom}}_{1122} = C^{\text{hom}}_{2211} &= (3k^s - 2\mu^s)\,\frac{\kappa_2 + \pi\left(\kappa_1 + \frac{8}{3}\epsilon\right)\left(1-\kappa_1\right)}{3\kappa_2 + 3\pi\kappa_1\left(1-\kappa_1\right) + 4\pi\epsilon}\\
C^{\text{hom}}_{1133} = C^{\text{hom}}_{2233} = C^{\text{hom}}_{3311} = C^{\text{hom}}_{3322} &= (3k^s - 2\mu^s)\,\frac{\kappa_2 + \pi\kappa_1\left(1-\kappa_1\right)}{3\kappa_2 + 3\pi\kappa_1\left(1-\kappa_1\right) + 4\pi\epsilon}\\
C^{\text{hom}}_{2323} = C^{\text{hom}}_{1313} &= \mu^s\,\frac{4\kappa_3 + \pi\left(1-\kappa_1\right)\left(1+2\kappa_1\right)}{4\kappa_3 + \frac{16}{3}\pi\epsilon\left(1-\kappa_1\right) + \pi\left(1+2\kappa_1\right)\left(1-\kappa_1\right)}\ ;\ C^{\text{hom}}_{1212} = \mu^s
\end{aligned} \tag{16}$$

where

$$\kappa_1 = \frac{3k^s + \mu^s}{3k^s + 4\mu^s}\ ;\ \kappa_2 = \frac{3k_n a}{3k^s + 4\mu^s}\ ;\ \kappa_3 = \frac{3k_t a}{3k^s + 4\mu^s}\ ;\ \kappa_4 = \frac{\mu^s}{3k^s + 4\mu^s}. \tag{17}$$

Such an estimate is given in Maghous et al. [7] and validated with finite element solutions based on the cohesive model (see Needleman [16]) by Maghous et al. [8].

We consider now the case of a material with two embedded families of microfractures, the orientation of each fracture family is defined in the 3D space by two spherical angular coordinates $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$. Although the fractures exhibit the same shape, their size can be different, which implies different properties. Therefore, two volume fractions (f_1, f_2) will be considered. Referring to eq. (13) (f_1, f_2) are related to aspect ratios (X_1, X_2), average radii (a_1, a_2) and fracture density parameters (ϵ_1, ϵ_2). The Mori-Tanaka estimate (eq. (14)) is given by:

$$\begin{aligned}
\mathbb{C}^{\text{hom}} = \lim_{X_1, X_2\to 0} \Bigg\{ &\left[\mathbb{C}^s + f_1\,\mathbb{C}^f_1 : \left(\mathbb{I} + \mathbb{P}_1 : \left(\mathbb{C}^f_1 - \mathbb{C}^s\right)\right)^{-1} + f_2\,\mathbb{C}^f_2 : \left(\mathbb{I} + \mathbb{P}_2 : \left(\mathbb{C}^f_2 - \mathbb{C}^s\right)\right)^{-1} \right] :\\
&\left[\mathbb{I} + f_1\left(\mathbb{I} + \mathbb{P}_1 : \left(\mathbb{C}^f_1 - \mathbb{C}^s\right)\right)^{-1} + f_2\left(\mathbb{I} + \mathbb{P}_2 : \left(\mathbb{C}^f_2 - \mathbb{C}^s\right)\right)^{-1} \right]^{-1} \Bigg\}
\end{aligned} \tag{18}$$

C^f_1 and C^f_2 refer to the stiffness of fracture families and are defined by the same format indicated by eq. (15) according to respective properties. We also introduce Hill tensor as $\mathbb{P}_1 = \mathbb{P}_1\left(X_1, \underline{n}_1\right)$ and $\mathbb{P}_2 = \mathbb{P}_2\left(X_2, \underline{n}_2\right)$. The global frame $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ is chosen as coincident with the local orientation $(\underline{t}, \underline{t}', \underline{n})$ of fracture family 1, making it necessary to perform the rotation only for the second family of fractures, i.e., in the terms that have index 2 in the eq. (18).

The estimate of the Mori-Tanaka scheme of eq. (18) for the particular case of spherical coordinates $(\theta, \varphi) = \left(\frac{\pi}{2}, 0\right)$ was given in homogenized flexibility $\mathbb{S}^{\text{hom}}$, since the size of the terms are smaller compared to equivalent stiffness. Therefore, each non-zero component of the equivalent elastic flexibility tensor is defined by eq. (19), remembering that the relationship $\mathbb{S}^{\text{hom}} = {\mathbb{C}^{\text{hom}}}^{-1}$ is valid.

$$
\begin{aligned}
S_{1111}^{\text{hom}} &= \frac{1}{(3k^s+4\mu^s)}\frac{12\pi\epsilon_2\left(\kappa_1-\kappa_4\right)\kappa_4+\left(3\pi\kappa_1\kappa_4+\tilde{\kappa}_2\right)\kappa_1}{3\kappa_4\left(\kappa_1-\kappa_4\right)\left(3\pi\kappa_1\kappa_4+\tilde{\kappa}_2\right)}\\
S_{1122}^{\text{hom}} = S_{2211}^{\text{hom}} = S_{1133}^{\text{hom}} = S_{3311}^{\text{hom}} = S_{2233}^{\text{hom}} = S_{3322}^{\text{hom}} &= -\frac{3k^s-2\mu^s}{18k^s\mu^s};\ S_{2222}^{\text{hom}} = \frac{3k^s+\mu^s}{9k^s\mu^s}\\
S_{3333}^{\text{hom}} &= \frac{1}{(3k^s+4\mu^s)}\frac{12\pi\epsilon_1\left(\kappa_1-\kappa_4\right)\kappa_4+\left(3\pi\kappa_1\kappa_4+\kappa_2\right)\kappa_1}{3\kappa_4\left(\kappa_1-\kappa_4\right)\left(3\pi\kappa_1\kappa_4+\kappa_2\right)}\\
S_{1212}^{\text{hom}} &= \frac{1}{\mu^s}\frac{4\tilde{\kappa}_3+\frac{16}{3}\pi\epsilon_2\left(1-\kappa_1\right)+\pi\left(1-\kappa_1\right)\left(1+2\kappa_1\right)}{4\tilde{\kappa}_3+\pi\left(1-\kappa_1\right)\left(1+2\kappa_1\right)}\\
S_{2323}^{\text{hom}} &= \frac{1}{\mu^s}\frac{4\kappa_3+\frac{16}{3}\pi\epsilon_1\left(1-\kappa_1\right)+\pi\left(1-\kappa_1\right)\left(1+2\kappa_1\right)}{4\kappa_3+\pi\left(1-\kappa_1\right)\left(1+2\kappa_1\right)}\\
S_{1313}^{\text{hom}} &= \frac{1}{\mu^s}\frac{\left[\pi^2\kappa_4\left(\epsilon_1+\epsilon_2\right)+\frac{9}{16}\pi^2\left(\kappa_1+\kappa_4\right)\kappa_4+\frac{\pi}{4}\left(\kappa_3+\tilde{\kappa}_3\right)\right]\left(\kappa_1+\kappa_4\right)+\frac{4}{9}\pi\left(\kappa_3\epsilon_2+\tilde{\kappa}_3\epsilon_1\right)+\frac{1}{9\kappa_4}\kappa_3\tilde{\kappa}_3}{\left[\frac{9}{16}\pi^2\left(\kappa_1+\kappa_4\right)\kappa_4+\frac{\pi}{4}\left(\kappa_3+\tilde{\kappa}_3\right)\right]\left(\kappa_1+\kappa_4\right)+\frac{1}{9\kappa_4}\kappa_3\tilde{\kappa}_3}
\end{aligned}
\tag{19}
$$

where $\tilde{\kappa}_2$ and $\tilde{\kappa}_3$ refer are similar to the terms defined in eq. (17), but relate to the parameters of the second family (k_{n2}, k_{t2}, a_2).

The last case analyzed refers to the isotropic distribution of short fractures (randomly oriented in the matrix), the orientation of each inclusion being defined in the 3D space by two spherical angular coordinates $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$. In this perspective, the effective stiffness tensor can be estimated by the following Mori-Tanaka scheme (eq. (20)), employing the same notation used in the case of parallel distributed fractures:

$$
\mathbb{C}^{\text{hom}} = \lim_{X\to 0}\left[\left(\mathbb{C}^s + \overline{\mathbb{C}^f : \left(\mathbb{I}+\mathbb{P}:\left(\mathbb{C}^f-\mathbb{C}^s\right)\right)^{-1}}\right) : \left(\mathbb{I} + \overline{\left(\mathbb{I}+\mathbb{P}:\left(\mathbb{C}^f-\mathbb{C}^s\right)\right)^{-1}}\right)^{-1}\right]
\tag{20}
$$

where the operator $\bar{\bullet}$ applied on a quantity $\mathcal{Q}$ denotes the integral on the spherical coordinates $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$ (see Maghous et al. [7]):

$$
\overline{\mathcal{Q}} = \int_0^{\pi} d\theta \int_0^{2\pi} \frac{4\pi}{3}\epsilon X \mathcal{Q}\left(\theta,\varphi\right)\frac{\sin\left(\theta\right)}{4\pi}d\varphi
\tag{21}
$$

The isotropic fracture distribution induces an isotropic homogeneous stiffness tensor, conveniently expressed by [4]:

$$
\mathbb{C}^{\text{hom}} = 3k^{\text{hom}}\mathbb{J} + 2\mu^{\text{hom}}\mathbb{K}
\tag{22}
$$

The homogenized bulk and shear moduli k^{hom} and μ^{hom} depend on the elastic properties of the constituents, the fracture radius a, and the fracture density parameter ϵ (Aguiar and Maghous [10]):

$$
k^{\text{hom}} = \frac{k^s}{1+\epsilon\mathcal{M}_k} \quad ; \quad \mu^{\text{hom}} = \frac{\mu^s}{1+\epsilon\mathcal{M}_\mu}
\tag{23}
$$

where dimensionless functions $\mathcal{M}_k\left(k^s, \mu^s, ak_n, ak_t\right)$ e $\mathcal{M}_\mu\left(k^s, \mu^s, ak_n, ak_t\right)$ are given by eq. (24):

$$
\mathcal{M}_k = \frac{4\pi\left(\kappa_1-\kappa_4\right)}{3\kappa_2+3\pi\kappa_1\left(1-\kappa_1\right)} \ \text{ e } \ \mathcal{M}_\mu = \frac{16\pi\kappa_4}{15}\frac{6\kappa_2+4\kappa_3+9\pi\kappa_4\left(3\kappa_1+\kappa_4\right)}{\left(3\pi\kappa_1\kappa_4+\kappa_2\right)\left[4\kappa_3+9\pi\kappa_4\left(\kappa_1+\kappa_4\right)\right]}
\tag{24}
$$

Note that for the described formulation it is necessary to know at least two parameters among $(\epsilon, \mathcal{N}, a)$ that quantify the damage. The particular case of cracked material (see Dormieux and Kondo [9]) is retrieved by setting $\underline{\underline{k}} = 0$ in the scheme of the $\mathbb{C}^{\text{hom}}$. In this situation, the average fracture radius a and the number of fractures per unit volume $\mathcal{N}$ are naturally eliminated from eq. (24), only being indirectly evaluated by means of ϵ which reduces the number of parameters that quantify the damage to just one.

4 Conclusions

In the present paper, the concepts of micromechanics applied to heterogeneous medium were presented in order to describe a formulation for the homogenized elastic behavior of fractured materials. In this context, using the tools of homogenization theory, especially the Mori-Tanaka scheme, the equivalent elastic properties of a fractured material were determined in three different situations. The first approach is related to a material with distribution of a single fracture family in the isotropic matrix and the results were determined through the non-zero components of the homogenized elastic stiffness tensor. Then the particular configuration of the two fracture families with perpendicular orientation was addressed. The two families exhibit distinct properties, which makes the formulation more comprehensive and the results were presented by means of the homogenized elastic flexibility tensor. It should also be emphasized that in the future we intend to present the results for the general case of two families, considering generic angular coordinates. The last approach refers to the isotropic case, in which the fractures are randomly distributed in the matrix and the results were demonstrated using the parameters k^{hom} and μ^{hom} which constitute the homogenized elastic stiffness tensor, also isotropic.

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