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Influence evaluation of high-order terms in the strain tensor for a complete geometric nonlinear analysis with Timoshenko element

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Abstract. This work evaluates the influence of high-order terms in the strain tensor associated to Timoshenko beam theory for a geometric nonlinear analysis. The tangent stiffness matrix of the studied element considers an updated Lagrangian formulation, shear deformation and the high-order terms in the strain tensor. A complete geometric nonlinear analysis with robust nonlinear solution schemes are performed for structures with a moderate slenderness ratio. The response is compared with a conventional updated Lagrangian formulation disregarding the high-order terms in the strain tensor, and with a corotational formulation. Examples evidence the importance of the high-order terms in strain tensor to perform geometric nonlinear analyses when considering an updated Lagrangian formulation. Moreover, the analysis with reduced element discretization, using the proposed formulation, provides equilibrium paths that are closer to highly discretized models compared to the others formulations.

Keywords: beam-column elements, Timoshenko theory, geometric stiffness matrix, post-critical behavior.

1 Introduction

Distinct aspects influence the behavior of a beam-column in a geometric nonlinear analysis using the Finite Element Method (FEM). The kinematic description of motion, the bending theory, the interpolation functions, and the strain tensor terms considered play an important role in these analyses.

The updated Lagrangian description of motion (McGuire [1], Bathe [2], Yang and Kuo [3]) leads to satisfactory results for the geometric nonlinear response of frames. In most works (Rodrigues et al. [4]), the formulation usually considers only Euler-Bernoulli beam theory. However, Rodrigues et al. [4] has shown the importance of considering the effects of shear deformation (Timoshenko beam theory) in the analysis of beam-column elements with a moderate slenderness ratio or with materials that have a small ratio of elastic to shear modulus. That study considered only the pre-critical phase.

Furthermore, Rodrigues et al. [4,5] also explored the strain tensor by using high-order terms to derive the local tangent stiffness matrix of a frame element considering the Timoshenko theory. While Rodrigues et al. [4] used cubic interpolation functions (as presented in Martha [6]), Rodrigues et al. [5] considered the influence of internal axial force and employed complete interpolation functions (Burgos and Martha [7]) to perform geometric nonlinear analyses with reduced structural discretization in the pre-critical phase.

The analysis of post-critical behavior of frames usually requires a refined discretization of the structure and a robust incremental-iterative method to solve the problem due to the possibility of complex equilibrium paths with critical points and multiple responses to a given load level (Rangel [8]). Thus, many authors work on the development of algorithms for this type of analysis (Leon et al. [9], Crisfield [10], Pacoste and Eriksson [11]). Rodrigues et al. [12] used these methods with a complete formulation of the stiffness matrix and reduced discretization, reaching an efficient behavior in the pre-critical phase and satisfactory results in the post-critical.

This research studies the element proposed in Rodrigues et al. [4] considering the updated Lagrangian formulation, the Timoshenko beam theory, and high-order terms in the strain tensor, in association to a robust incremental-iterative method to explore how these aspects influence the post-critical behavior of geometric nonlinear analyses of framed structures. The computational implementations were made in the NUMA-TF (*NUMerical Analysis of Truss and Frames*) program [13], an open-source MATLAB program.

2 Element formulation

The element studied was formulated in Rodrigues et al. [4]. This element considers high-order terms in the strain tensor of a Timoshenko beam-column with an updated Lagrangian description. The geometric stiffness matrix was derived using the interpolation functions of a Timoshenko element and the principle of virtual work.

2.1 Timoshenko beam theory (TBT)

In the Timoshenko beam theory (TBT), the shear distortion (γ) is an additional cross-section rotation, so the transverse displacement is decoupled from the cross-section rotation, as shown in Fig. 1. The displacement field is written according to eq. (1).

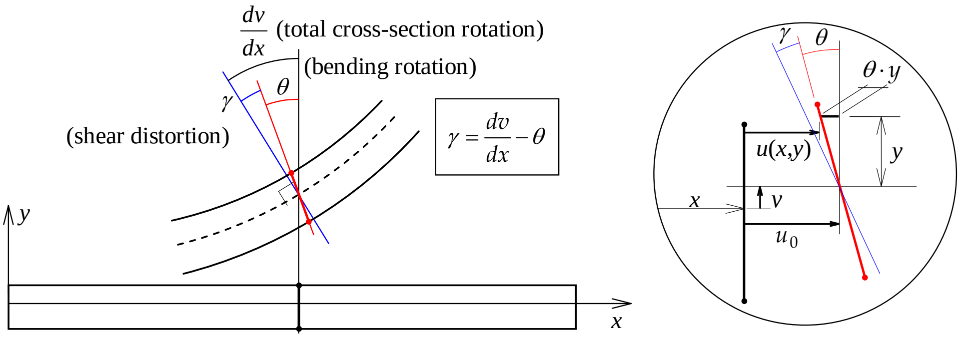

Figure 1. Timoshenko beam displacement field (Rodrigues et al. [5])

$$u(x, y) = u_0(x) - \theta(x).y \qquad v(x, y) = v_0(x) \tag{1}$$

The equilibrium of an infinitesimal element is depicted in Fig. 2, and the differential equation of the problem is given in eq. (2). Equation (3) can be written by replacing the shear force acting on a section (Q), where G is the shear modulus, A is the cross-section area, and χ is the factor that defines the effective shear area of the cross-section.

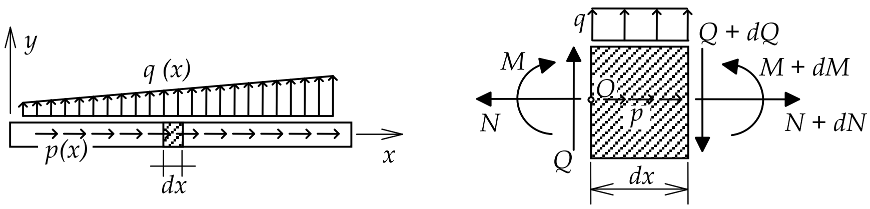

Figure 2. Timoshenko beam displacement field (Rodrigues et al. [4])

$$\sum M_0 = 0 \rightarrow dM(x)/dx = Q(x) \rightarrow EI(d^2\theta/dx^2) - Q(x) = 0. \tag{2}$$

$$EI(d^2\theta/dx^2) + \chi GA[dv(x)/dx - \theta(x)] = 0. \tag{3}$$

Using the dimensionless factor Ω, introduced by Reddy [14], the homogenous solution of the differential equation is given by eq. (4).

$$
\begin{gathered}
dv(x)/dx = \theta(x) + \gamma(x) \\
v_0^h(x) = c_0 + c_1 x + c_2 \frac{x^2}{2} + c_3\left(\frac{x^3}{6} - \Omega L^2 x\right) \\
\theta_0^h(x) = c_1 + c_2 x + c_3 \frac{x^2}{2}
\end{gathered}
\quad , \quad \Omega = \frac{EI}{\chi GA}\frac{1}{L^2} \tag{4}
$$

This solution can be used to write interpolating functions of nodal transverse displacements and rotations:

$$
\begin{aligned}
N_2^v(x) &= 1 + \frac{2\left(\frac{x}{L}\right)^3 - 3\left(\frac{x}{L}\right)^2 - 12\Omega\frac{x}{L}}{(1+12\Omega)} & N_2^\theta(x) &= \frac{6}{L}\frac{\left(\frac{x}{L}\right)^2 - \frac{x}{L}}{(1+12\Omega)} \\
N_3^v(x) &= \frac{x\left[\left(\frac{x}{L}\right)^2 - 2(1+3\Omega)\frac{x}{L} + 1 + 6\Omega\right]}{(1+12\Omega)} & N_3^\theta(x) &= 1 + \frac{3\left(\frac{x}{L}\right)^2 - 4(1+3\Omega)\frac{x}{L}}{(1+12\Omega)} \\
N_5^v(x) &= \frac{3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3 + 12\Omega\frac{x}{L}}{(1+12\Omega)} & N_5^\theta(x) &= \frac{6}{L}\frac{\frac{x}{L} - \left(\frac{x}{L}\right)^2}{(1+12\Omega)} \\
N_6^v(x) &= \frac{x\left[\left(\frac{x}{L}\right)^2 - (1+6\Omega)\frac{x}{L} - 6\Omega\right]}{(1+12\Omega)} & N_6^\theta(x) &= \frac{3\left(\frac{x}{L}\right)^2 - 2(1-6\Omega)\frac{x}{L}}{(1+12\Omega)}
\end{aligned} \tag{5}
$$

2.2 Updated Lagrangian formulation

The local stiffness matrix is developed in Rodrigues et al. [4]. Considering an updated Lagrangian formulation (McGuire et al. [1], Yang and Kuo [3]), the virtual work of the problem is written as in eq. (6).

$$
\int_V C_{ijkl}\Delta e_{kl}\delta\Delta e_{ij}\, dV + \int_V \tau_{ij}^t \delta\Delta e_{ij}\, dV + \int_V \tau_{ij}^t\, \Delta\eta_{ij} dV = R^{(t+\Delta t)} \tag{6}
$$

The first integral leads to the elastic stiffness matrix, while the third integral leads to the geometric stiffness matrix. The second integral represents the virtual work of forces acting on the element in configuration *t* and is usually presented on the right-hand side of the expression (Rodrigues et al. [5]). Typically, the higher-order terms are disregarded in the Green-Lagrange strain tensor, which is written as in eq. (8), leading to the reduced stiffness matrix presented in eq. (9).

$$
\eta_{xx} = \frac{1}{2}\left(\frac{\partial u}{\partial x}^2 + \frac{\partial v}{\partial x}^2\right) \quad \eta_{xy} = 0 \tag{8}
$$

$$
K^g = \begin{bmatrix}
\frac{P}{L} & 0 & 0 & -\frac{P}{L} & 0 & 0 \\
0 & \frac{6P\left(120\Omega_Y^2 + 20\Omega_Y + 1\right)}{5L\left(12\Omega_Y + 1\right)^2} & \frac{P}{10\left(12\Omega_Y + 1\right)^2} & 0 & -\frac{6P\left(120\Omega_Y^2 + 20\Omega_Y + 1\right)}{5L\left(12\Omega_Y + 1\right)^2} & \frac{P}{10\left(12\Omega_Y + 1\right)^2} \\
0 & \frac{P}{10\left(12\Omega_Y + 1\right)^2} & \frac{2LP\left(90\Omega_Y^2 + 15\Omega_Y + 1\right)}{15\left(12\Omega_Y + 1\right)^2} & 0 & -\frac{P}{10\left(12\Omega_Y + 1\right)^2} & -\frac{LP\left(360\Omega_Y^2 + 60\Omega_Y + 1\right)}{30\left(12\Omega_Y + 1\right)^2} \\
-\frac{P}{L} & 0 & 0 & \frac{P}{L} & 0 & 0 \\
0 & -\frac{6P\left(120\Omega_Y^2 + 20\Omega_Y + 1\right)}{5L\left(12\Omega_Y + 1\right)^2} & -\frac{P}{10\left(12\Omega_Y + 1\right)^2} & 0 & \frac{6P\left(120\Omega_Y^2 + 20\Omega_Y + 1\right)}{5L\left(12\Omega_Y + 1\right)^2} & -\frac{P}{10\left(12\Omega_Y + 1\right)^2} \\
0 & \frac{P}{10\left(12\Omega_Y + 1\right)^2} & -\frac{LP\left(360\Omega_Y^2 + 60\Omega_Y + 1\right)}{30\left(12\Omega_Y + 1\right)^2} & 0 & -\frac{P}{10\left(12\Omega_Y + 1\right)^2} & \frac{2LP\left(90\Omega_Y^2 + 15\Omega_Y + 1\right)}{15\left(12\Omega_Y + 1\right)^2}
\end{bmatrix} \tag{9}
$$

To derive the geometric stiffness matrix taking into account the higher-order terms in the strain tensor, Rodrigues et al. [4] considered the nonlinear part of the Green-Lagrange strain tensor ($\Delta\eta$) associated with the displacements field of the Timoshenko beam, provided eq. (1), leading to eq. (10).

$$
\eta_{xx} = \frac{1}{2}\left(\frac{\partial u}{\partial x}^2 + \frac{\partial v}{\partial x}^2\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x}^2 + \frac{\partial v}{\partial x}^2 + y^2\frac{\partial \theta_z}{\partial x}^2\right) - y\frac{\partial u}{\partial x}\frac{\partial \theta_z}{\partial x} \ , \quad \eta_{xy} = \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = y\frac{\partial \theta_z}{\partial x}\theta_z - \theta_z\frac{\partial u}{\partial x} \tag{10}
$$

Therefore, by using the interpolation functions of eq. (5), the virtual work of eq. (6), and the nonlinear part of the Green-Lagrange strain tensor of eq. (10), Rodrigues et al. [4] reached the geometric stiffness matrix presented in eq. (11).

$$K^g = \begin{bmatrix} \frac{P}{L} & 0 & -\frac{M_{Z1}}{L} & -\frac{P}{L} & 0 & -\frac{M_{Z2}}{L} \\ 0 & \frac{6P\gamma}{5L\mu^2}+\frac{12PI_Z}{AL^3\mu^2} & \frac{P}{10\mu^2}+\frac{6PI_Z}{AL^2\mu^2} & 0 & -\frac{6P\gamma}{5L\mu^2}-\frac{12PI_Z}{AL^3\mu^2} & \frac{P}{10\mu^2}+\frac{6PI_Z}{AL^2\mu^2} \\ -\frac{M_{z1}}{L} & \frac{P}{10\mu^2}+\frac{6PI_Z}{AL^2\mu^2} & \frac{2LP\phi}{15\mu^2}+\frac{4PI_Z\alpha}{AL\mu^2} & \frac{M_{Z1}}{L} & -\frac{P}{10\mu^2}-\frac{6PI_Z}{AL^2\mu^2} & -\frac{LP\eta}{30\mu^2}-\frac{2PI_Z\rho}{AL\mu^2} \\ -\frac{P}{L} & 0 & \frac{M_{Z1}}{L} & \frac{P}{L} & 0 & \frac{M_{z2}}{L} \\ 0 & -\frac{6P\gamma}{5L\mu^2}-\frac{12PI_Z}{AL^3\mu^2} & -\frac{P}{10\mu^2}-\frac{6PI_Z}{AL^2\mu^2} & 0 & \frac{6P\gamma}{5L\mu^2}+\frac{12PI_Z}{AL^3\mu^2} & -\frac{P}{10\mu^2}-\frac{6PI_Z}{AL^2\mu^2} \\ -\frac{M_{z2}}{L} & \frac{P}{10\mu^2}+\frac{6PI_Z}{AL^2\mu^2} & -\frac{LP\eta}{30\mu^2}-\frac{2PI_Z\rho}{AL\mu^2} & \frac{M_{z2}}{L} & -\frac{P}{10\mu^2}-\frac{6PI_Z}{AL^2\mu^2} & \frac{2LP\phi}{15\mu^2}+\frac{4PI_Z\alpha}{AL\mu^2} \end{bmatrix} \tag{11}$$

$$\mu = (12\Omega_Y + 1) \qquad \gamma = (120\Omega_Y{}^2 + 20\Omega_Y + 1) \qquad \phi = (90\Omega_Y{}^2 + 15\Omega_Y + 1)$$
$$\alpha = (36\Omega_Y{}^2 + 6\Omega_Y + 1) \qquad \eta = 360\Omega_Y{}^2 + 60\Omega_Y + 1 \qquad \rho = 72\Omega_Y{}^2 + 12\Omega_Y - 1$$

Finally, the internal forces are calculated considering large displacements. According to McGuire et al. [1], the rigid body motions should be separated from the natural deformations (Fig. 3). The displacements are limited to axial displacement (u_n) and nodal rotations (θ_{an}, θ_{bn}). By employing robust solution schemes to solve the nonlinear problem, Rodrigues et al. [12] explained that the forces at the end of a load step $\{^2F\}$ are calculated considering the forces at the beginning of the step $\{^1F\}$ and their increment {dF} up to the current iteration: $\{^2F\} = \{^1F\} + \{dF\}$. The increment of forces should be calculated with the tangent matrix, as in eq. (12).

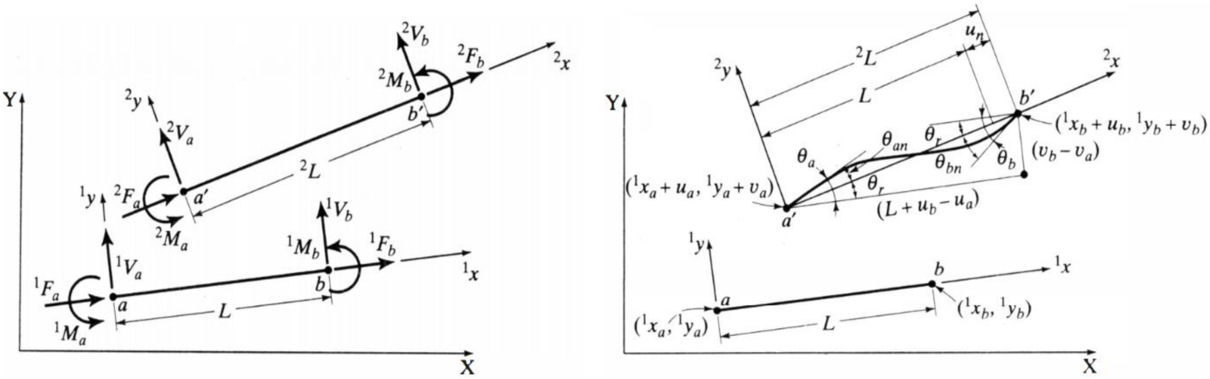

Figure 3. Element forces and displacements (McGuire *et al.* [1])

$$\{dF\} = [K_e + K_g][0 \;\; 0 \;\; \theta_{an} \;\; u_n \;\; 0 \;\; \theta_{bn}]^T \tag{12}$$

2.3 Corotational formulation

The corotational formulation is well presented in Crisfield [10] and Rangel [8]. The tangent stiffness matrix in the global system is calculated as follows:

$$[K] = [T][K_e][T]^T + {}^{P}/_{L}\{z\}\{z\}^T + {}^{M_{z1}+M_{z2}}/_{L^2}(\{z\}\{r\}^T + \{r\}\{z\}^T) \tag{13}$$

where $[T]$, $\{z\}$ and $\{r\}$ are transformation matrices and vectors. The natural system matrix, K_e, corresponds to the elastic matrix, which is given by eq. (14) when considering the Timoshenko theory.

$$K_e = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI_Z}{L^3(12\Omega_Y+1)} & \frac{6EI_Z}{L^2(12\Omega_Y+1)} & 0 & -\frac{12EI_Z}{L^3(12\Omega_Y+1)} & \frac{6EI_Z}{L^2(12\Omega+1)} \\ 0 & \frac{6EI_Z}{L^2(12\Omega_Y+1)} & \frac{4EI_Z(3\Omega_Y+1)}{L(12\Omega_Y+1)} & 0 & -\frac{6EI_Z}{L^2(12\Omega_Y+1)} & \frac{2EI_Z(1-6\Omega_Y)}{L(12\Omega_Y+1)} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI_Z}{L^3(12\Omega_Y+1)} & -\frac{6EI_Z}{L^2(12\Omega_Y+1)} & 0 & \frac{12EI_Z}{L^3(12\Omega_Y+1)} & -\frac{6EI_Z}{L^2(12\Omega_Y+1)} \\ 0 & \frac{6EI_Z}{L^2(12\Omega_Y+1)} & \frac{2EI_Z(1-6\Omega_Y)}{L(12\Omega_Y+1)} & 0 & -\frac{6EI_Z}{L^2(12\Omega_Y+1)} & \frac{4EI_Z(3\Omega_Y+1)}{L(12\Omega_Y+1)} \end{bmatrix} \tag{14}$$

3 Numerical analysis

The results obtained with the proposed Timoshenko Large formulation (higher-order terms in the strain tensor – TBT-Large) were compared with other formulations considering different discretization levels: conventional Timoshenko theory (TBT-Small), Euler-Bernoulli theory considering higher-order terms in the strain tensor (EBBT-Large), corotational formulation (TBT-Corotational). The influence of higher-order terms in the strain tensor considering the Timoshenko beam theory (TBT) was studied with three models (Fig. 4): a Toggle frame, a Roorda, and a Lee frame. In the first two, the structure was discretized with 2 and 5 elements per bar, and only 5 for the Lee frame due to its higher nonlinearity. Moreover, the corotational formulation with a high discretization of 10 elements per bar was taken as a reference solution for all models. Examples consider a length of L = 1 m, Young's modulus of $E = 10^7$ kN/m^2, Poisson's ratio of ν = 0.3, cross-section form factor of χ = 5/6, and reduced slenderness ratio of L/h = 4. The resulting equilibrium paths are shown in Fig. 5 to Fig. 9.

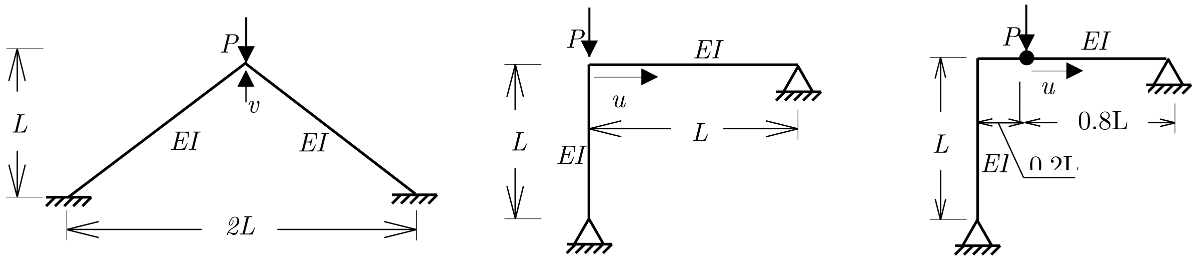

Figure 4. (a) Toggle frame, (b) Roorda frame, (c) Lee frame

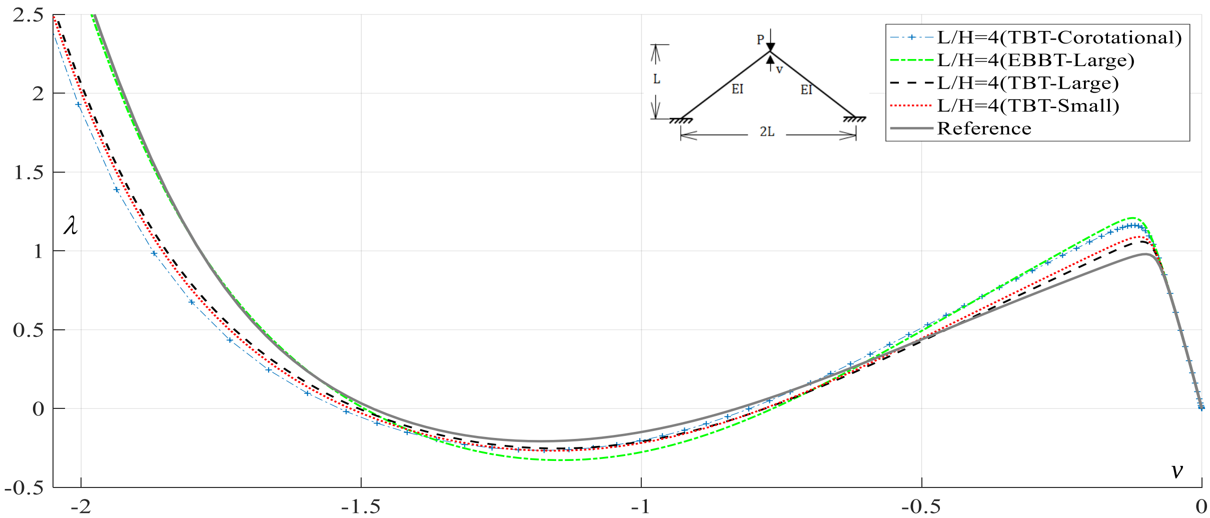

Figure 5. Toggle frame equilibrium paths for 2 elements in each bar - L/h = 4

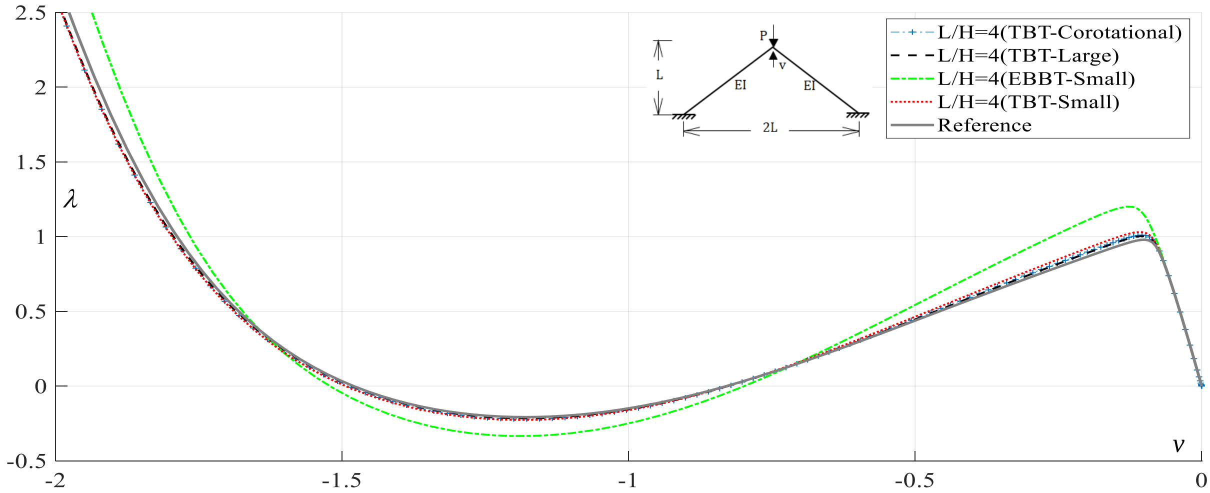

Figure 6. Toggle frame equilibrium paths for 5 elements in each bar - L/h = 4

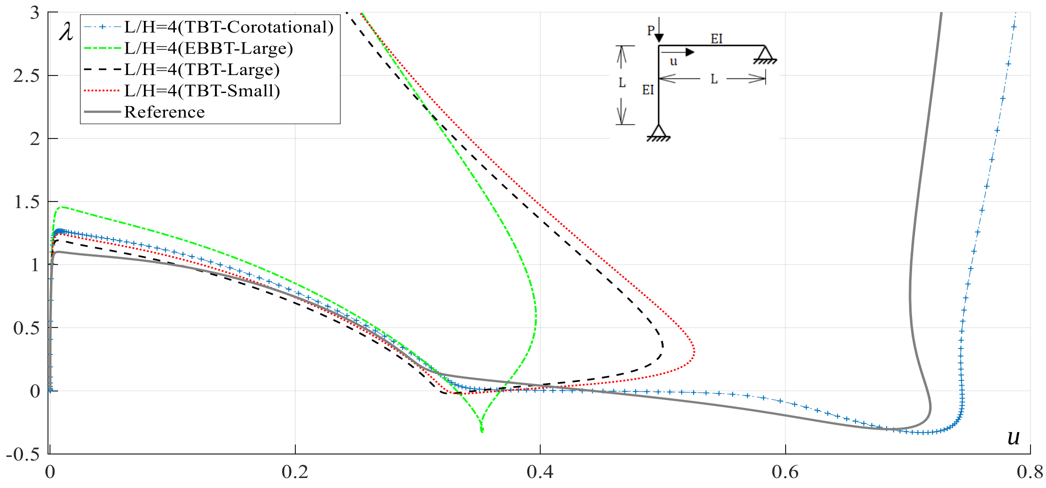

Figure 7. Roorda frame equilibrium paths for 2 elements in each bar - L/h = 4

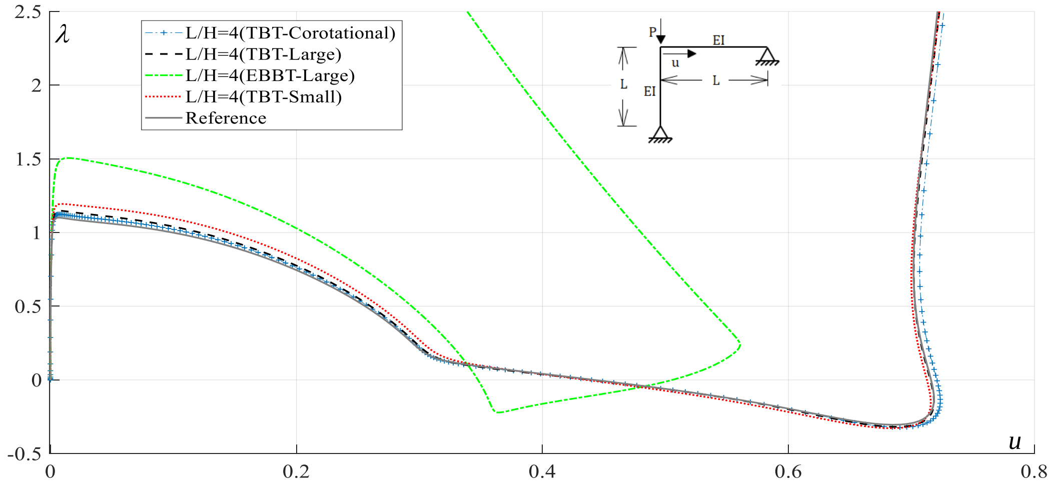

Figure 8. Roorda frame equilibrium paths for 5 elements in each bar - L/h = 4

It can be observed that considering only 2 elements per bar in the Toggle and Roorda frames, the equilibrium paths overlap in the pre-critical phase and the buckling load is slightly better approximated with the TBT_Large formulation. In the first example, the buckling load predicted by the corotational formulation is similar to the EBBT elements. Meanwhile, for the discretization with 5 elements per bar, all curves of the Timoshenko theory approach the reference response. In the Toggle frame, the Large formulation provides a slightly better agreement. In the Roorda frame, the corotational formulation has the best agreement in the beginning of the post-critical phase, followed closely by the Large formulation, which approaches the reference solution better in the end of the post-critical phase.

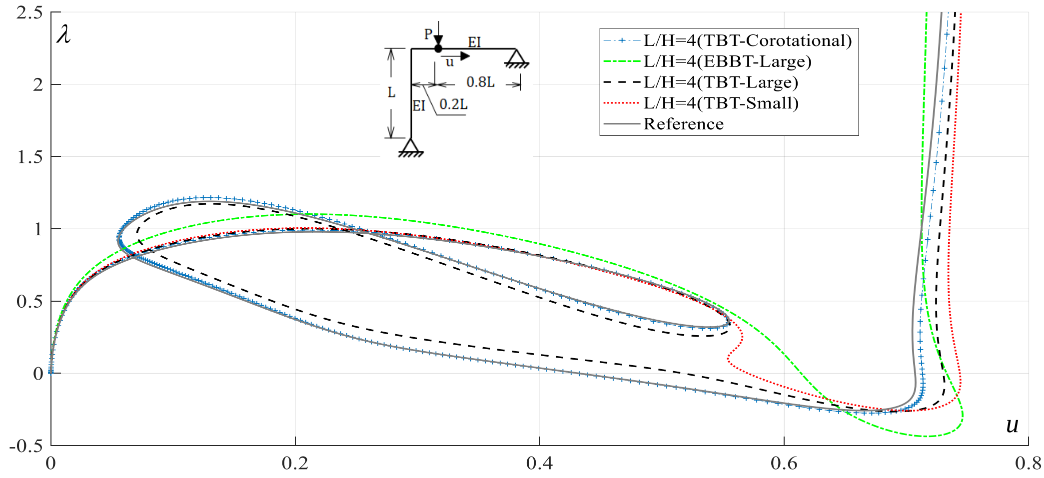

Figure 9. Lee frame equilibrium path for 5 elements in each bar - L/h = 4

The equilibrium paths of the Lee frame indicate that the consideration of higher-order terms in the strain tensor (TBT_Large) leads to similar results to other formulations during the pre-critical phase and in the beginning of the post-critical phase. However, when this effect is disregarded (TBT-Small), there is a point in the post-critical phase in which the solution cannot follow the reference result, generating a wrong behavior. The corotational formulation provides the equilibrium path with the best agreement with respect to the reference response in the post-critical phase, and it is well followed by the Large formulation. Finally, the example shows that assuming the Euler-Bernoulli beam theory for structures with small slenderness provides incorrect results.

4 Conclusions

The results showed the influence of higher-order terms in the strain tensor and the assumption of Timoshenko beam theory. For small slenderness ratios, it is fundamental to consider the shear deformation during the bending of elements, and it is also important to take into account the high-order terms in the strain tensor. The post-critical behavior is clearly affected if a Large or a Small formulation is employed and if they are associated with Euler-Bernoulli or Timoshenko beam theory. The consideration of higher-order terms approximates the equilibrium paths to the reference response of a highly discretized structure in pre-critical and post-critical phases. When using a reduced discretization, the consideration of high-order terms leads to an equilibrium path with better approximation of the buckling load and better agreement with reference response in the beginning of the post-critical phase. The corotational formulation is better approximated by the updated Lagrangian formulation with higher-order terms in the strain tensor especially in the post-critical phase.

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