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A comparison of the effects of nonlinear damping on the free vibration of laminated circular cylindrical shells

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Abstract. Knowing the damping level is important for the analysis, experimentation, and use of the systems. Most of laminated shells studies are concerned with viscous damping, however in several engineering applications the nonlinear damping is introduced by dissipative forces. For this reason, in this study the influence of the nonlinear damping term proportional to the power of velocity on the free vibration of laminated circular cylindrical shells is considered. To model the shell, the Donnell nonlinear shallow shell theory is used, and from energy approaches the set of nonlinear ordinary differential equations of motion is obtained, and then solved by Runge-Kutta method. For these analyzes is considered expansions to describe the axial, circumferential and radial displacements totalizing fourteen degrees-of-freedom. The obtained results show that nonlinear damping have a great influence on the attenuation of free vibration of the laminated circular cylindrical shells.

Keywords: structural nonlinear damping, laminated cylindrical shell, free vibration.

1 Introduction

Knowing the damping level is important for the analysis, experimentation, and use of the systems. It acts in the attenuation of the resonance response in forced systems or in the decay of free vibrations [1, 2, 3], and in various systems the energy dissipation mechanism is nonlinear [4].

Often, damping is the main responsible for the nonlinear behavior of the system and, therefore, it is interesting that the responses are analyzed assuming only this portion as nonlinear [5]. The nonlinearity condition in damping may presents a rich variety of behaviors as: coexistence of attractors, coexistence of an attractor and a sea of chaos, chaotic responses without equilibrium points, pairs of attractors and repellors [6].

Most of laminated shells studies are concerned with viscous damping, however in several engineering applications the nonlinear damping is introduced by dissipative forces. For this reason, in this work the influence of nonlinear damping term proportional to the power of velocity on the free vibration behavior of laminated circular cylindrical shells is investigated.

To model the shell, the Donnell nonlinear shallow shell theory without considering the effect of shear deformation is used, and fourteen degrees-of-freedom is considered to describe the displacement field in axial, circumferential and radial directions. The set of coupled nonlinear ordinary differential equations of motion are derived from energy approaches and, in turn, solved by Runge-Kutta method. The obtained results show that nonlinear damping have a great influence on the free vibration responses of the laminated circular cylindrical shells.

2 Mathematical formulation

Consider a simply supported laminated cylindrical shell with length L, radius R and thickness h, made of N orthotropic layers. The axial, circumferential and radial global coordinates are denoted by x, $y = R\theta$ and z, respectively, and the corresponding displacements of the shell middle surface are denoted by u, v and w, as displayed in Fig. 1. Local coordinate system, which determines the principal axes of material orthotropy and may not coincides with the global cylindrical coordinates, are represented by α_1^k, α_2^k and α_3^k.

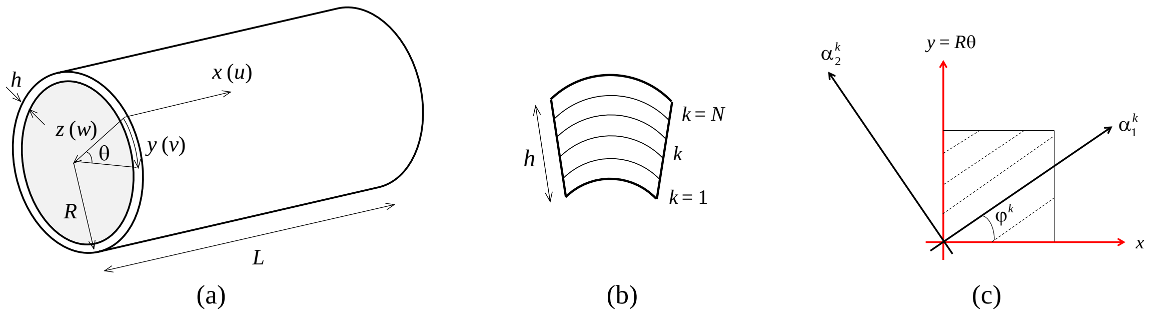

Figure 1. (a) Shell geometry, (b) Shell lamination and (c) Coordinate system of laminae

The kth layer is assumed to be made of an elastic orthotropic material with Young's moduli E_1^k and E_2^k in the α_1^k and α_2^k directions, respectively, shear modulus G_{12}^k, Poisson coefficients ν_{12}^k and ν_{21}^k, and mass density ρ_s^k. The stress-strain relation for the kth orthotropic layer of the shell in the local coordinates, obtained under the hypothesis $\bar{\sigma}_3^k = \bar{\tau}_{13}^k = \bar{\tau}_{23}^k = 0$, is given by

$$\begin{Bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \\ \bar{\tau}_{12} \end{Bmatrix}^k = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix}^k \begin{Bmatrix} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_2 \\ \bar{\gamma}_{12} \end{Bmatrix}, \tag{1}$$

where the coefficients C_{ij}^k are

$$C_{11}^k = \frac{E_1^k}{1-\nu_{12}^k\,\nu_{21}^k}, \qquad C_{22}^k = \frac{E_2^k}{1-\nu_{12}^k\,\nu_{21}^k}, \qquad C_{33}^k = G_{12}^k, \qquad C_{12}^k = C_{21}^k = \frac{\nu_{12}^k\,E_2^k}{1-\nu_{12}^k\,\nu_{21}^k}, \tag{2}$$

where because of symmetry, $\nu_{21}^k\,E_1^k = \nu_{12}^k\,E_2^k$.

Usually, the lamina material axes (α_1^k and α_2^k) do not coincide with the shell reference axes (x, θ), while α_3^k is normally coincident with z. Then, the strains and stresses on local axes can be related to the global axes by using the expressions

$$\begin{Bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \\ \bar{\tau}_{12} \end{Bmatrix}^k = \mathbf{T}_1^k \begin{Bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_\theta \\ \bar{\tau}_{x\theta} \end{Bmatrix}^k, \qquad \begin{Bmatrix} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_2 \\ \bar{\gamma}_{12} \end{Bmatrix} = \mathbf{T}_2^k \begin{Bmatrix} \bar{\varepsilon}_x \\ \bar{\varepsilon}_\theta \\ \bar{\gamma}_{x\theta} \end{Bmatrix}, \tag{3}$$

where

$$\begin{aligned} \mathbf{T}_1^k &= \begin{bmatrix} \cos^2\phi^k & \operatorname{sen}^2\phi^k & 2\operatorname{sen}\phi^k\cos\phi^k \\ \operatorname{sen}^2\phi^k & \cos^2\phi^k & -2\operatorname{sen}\phi^k\cos\phi^k \\ -\operatorname{sen}\phi^k\cos\phi^k & \operatorname{sen}\phi^k\cos\phi^k & \cos^2\phi^k-\operatorname{sen}^2\phi^k \end{bmatrix}, \\ \mathbf{T}_2^k &= \begin{bmatrix} \cos^2\phi^k & \operatorname{sen}^2\phi^k & \operatorname{sen}\phi^k\cos\phi^k \\ \operatorname{sen}^2\phi^k & \cos^2\phi^k & -\operatorname{sen}\phi^k\cos\phi^k \\ -2\operatorname{sen}\phi^k\cos\phi^k & 2\operatorname{sen}\phi^k\cos\phi^k & \cos^2\phi^k-\operatorname{sen}^2\phi^k \end{bmatrix}, \end{aligned} \tag{4}$$

and ϕ^k is the angle between x and α_1^k.

Therefore, the stress-strain relation in the shell coordinates can be written as

$$\begin{Bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_\theta \\ \bar{\tau}_{x\theta} \end{Bmatrix}^k = \mathbf{Q}^k \begin{Bmatrix} \bar{\varepsilon}_x \\ \bar{\varepsilon}_\theta \\ \bar{\gamma}_{x\theta} \end{Bmatrix}, \tag{5}$$

where the stiffness matrix $\mathbf{Q}^k$ is given by

$$\mathbf{Q}^k = \left(\mathbf{T}_1^k\right)^{-1} \mathbf{C}^k \, \mathbf{T}_2^k. \tag{6}$$

It is possible to observe, as a consequence of the discontinuous variation of the stiffness matrix from layer to layer, discontinuities in the stresses distribution.

The middle surface kinematic relations, based on the Donnell shallow-shell theory, are

$$\begin{gathered} \varepsilon_x = u_{,x} + \frac{1}{2} w_{,x}^2, \qquad \varepsilon_\theta = \frac{v_{,\theta}}{R} - \frac{w}{R} + \frac{1}{2}\frac{w_{,\theta}^2}{R^2}, \qquad \gamma_{x\theta} = \frac{u_{,\theta}}{R} + v_{,x} + w_{,x}\frac{w_{,\theta}}{R}, \\ \chi_x = -w_{,xx}, \qquad \chi_\theta = -\frac{w_{,\theta\theta}}{R^2}, \qquad \chi_{x\theta} = -\frac{w_{,x\theta}}{R}, \end{gathered} \tag{7}$$

which are related to the deformations of any point on the shell by

$$\begin{Bmatrix} \bar{\varepsilon}_x \\ \bar{\varepsilon}_\theta \\ \bar{\gamma}_{x\theta} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_\theta \\ \gamma_{x\theta} \end{Bmatrix} + z \begin{Bmatrix} \chi_x \\ \chi_\theta \\ 2\chi_{x\theta} \end{Bmatrix}. \tag{8}$$

The nonlinear equations of motion are obtained by applying Hamilton's principle to non-conservative systems. The kinetic energy is considered as

$$T = \int_0^L \int_0^{2\pi} \sum_{k=1}^{N} \left\{ \int_{h_k}^{h_{k+1}} \left[\frac{1}{2} \rho_s^k \left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2 \right) \right] R \, dz \right\} d\theta \, dx, \tag{9}$$

and the strain energy is given by

$$U = \int_0^L \int_0^{2\pi} \sum_{k=1}^{N} \left\{ \int_{h_k}^{h_{k+1}} \left[\frac{1}{2} \left(\bar{\sigma}_x^k \, \bar{\varepsilon}_x + \bar{\sigma}_\theta^k \, \bar{\varepsilon}_\theta + \bar{\tau}_{x\theta}^k \, \bar{\gamma}_{x\theta} \right) \left(1 + \frac{z}{R} \right) \right] R \, dz \right\} d\theta \, dx. \tag{10}$$

For the viscous damping is considered the Rayleigh’s dissipation function

$$F_1 = \int_0^L \int_0^{2\pi} \left[\frac{1}{2} \beta_1 \left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2 \right) \right] R \, d\theta \, dx, \tag{11}$$

where

$$\beta_1 = 2\,\zeta \sum_{k=1}^{N} \left[\rho_s^k \left(h_{k+1} - h_k \right) \right] \omega_0, \tag{12}$$

where ζ is the viscous damping coefficient and ω_0 is the lowest natural frequency of the shell.

Rayleigh dissipation function (eq. 8) is considered to obtain the linear damping; however, to obtain the quadratic and cubic damping, it is necessary to define equivalent dissipation functions, determined respectively as

$$F_2 = \int_0^L \int_0^{2\pi} \left[\frac{1}{3} \beta_2 \left(\dot{u}^3 \, \mathrm{signum}(\dot{u}) + \dot{v}^3 \mathrm{signum}(\dot{v}) + \dot{w}^3 \mathrm{signum}(\dot{w}) \right) \right] R \, d\theta \, dx, \tag{13}$$

$$F_3 = \int_0^L \int_0^{2\pi} \left[\frac{1}{4} \beta_3 \left(\dot{u}^4 + \dot{v}^4 + \dot{w}^4 \right) \right] R \, d\theta \, dx, \tag{14}$$

where β_2 and β_3 are quadratic and cubic damping coefficients.

For the displacements field the following modal expansions, in terms of the circumferential and axial variables, are adopted

$$
\begin{aligned}
u(x,\theta,t) &= \xi_{u_{1,1}}(t)\, h\cos(n\theta)\cos\left(\frac{m\pi x}{L}\right) + \xi_{u_{1,1c}}(t)\, h\,\mathrm{sen}(n\theta)\cos\left(\frac{m\pi x}{L}\right) \\
&\quad + \xi_{u_{0,1}}(t)\, h\cos\left(\frac{m\pi x}{L}\right) + \xi_{u_{0,3}}(t)\, h\cos\left(\frac{3m\pi x}{L}\right), \\
v(x,\theta,t) &= \xi_{v_{1,1}}(t)\, h\sin(n\theta)\sin\left(\frac{m\pi x}{L}\right) + \xi_{v_{1,1c}}(t)\, h\cos(n\theta)\sin\left(\frac{m\pi x}{L}\right) \\
&\quad + \xi_{v_{2,1}}(t)\, h\sin(2n\theta)\sin\left(\frac{m\pi x}{L}\right) + \xi_{v_{2,1c}}(t)\, h\cos(2n\theta)\sin\left(\frac{m\pi x}{L}\right) \\
&\quad + \xi_{v_{2,3}}(t)\, h\sin(2n\theta)\sin\left(\frac{3m\pi x}{L}\right) + \xi_{v_{2,3c}}(t)\, h\cos(2n\theta)\sin\left(\frac{3m\pi x}{L}\right), \\
w(x,\theta,t) &= \xi_{w_{1,1}}(t)\, h\cos(n\theta)\sin\left(\frac{m\pi x}{L}\right) + \xi_{w_{1,1c}}(t)\, h\sin(n\theta)\sin\left(\frac{m\pi x}{L}\right) \\
&\quad + \xi_{w_{0,1}}(t)\, h\sin\left(\frac{m\pi x}{L}\right) + \xi_{w_{0,3}}(t)\, h\sin\left(\frac{3m\pi x}{L}\right),
\end{aligned}
\tag{15}
$$

where $\xi_u(t)$, $\xi_v(t)$ e $\xi_w(t)$ are the time-dependent non-dimensional modal amplitudes.

These modal expansions satisfy the out-of-plane boundary conditions and include driven modes, companion modes, additional asymmetric modes, and axisymmetric modes. They lead to a fourteen-degrees-of-freedom reduced order model and, despite being minimal expansions, offer good accuracy [7].

3 Numerical results

The analysis has been made for a simply-supported, imperfection-free, laminated circular cylindrical shell with L = 95.87 mm, R = 67.80 mm and h = 0.678 mm, made of three layers -30°/0°/30° of the same thickness. The material properties of the three layers are $E_1 = 40.2 \times 10^9$ Pa, $E_2 = 16.7 \times 10^9$ Pa, $G_{12} = 4.61 \times 10^9$ Pa, $\nu_{12} = 0.363$ and ρ_s = 1500 kg/m³ [8]. In the present analysis, the adopted viscous damping factor is $\zeta = 0.01$, and the quadratic and cubic damping coefficients are considered as portions of linear damping coefficient (eq. 12).

The relations between quadratic or cubic and linear damping coefficients assume eight different values: 0; 1/10; 1; 10; 100; 1,000; 10,000; and 100,000. These values were chosen after carrying out an analysis of the individual influence of each damping coefficient on the free vibration behavior of the shell, displayed in Fig. 2.

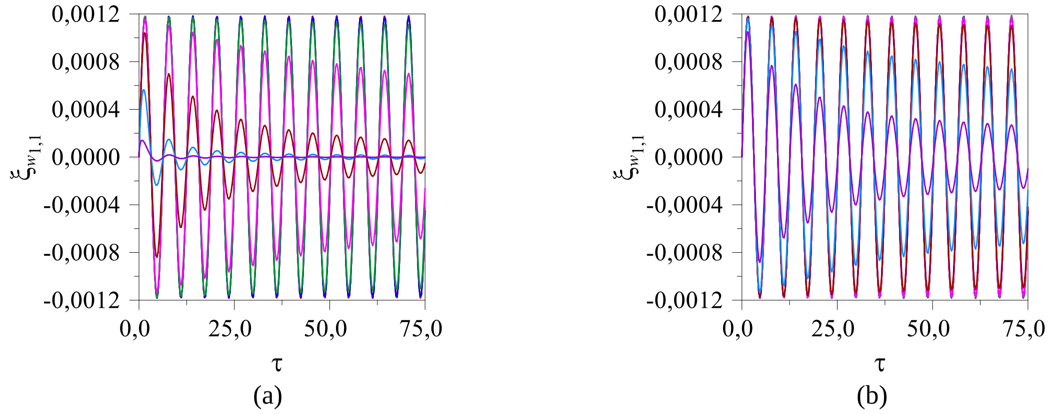

Figure 2. Free vibration response with $\beta_1 = 0$. (a) $\beta_3 = 0$ and (b) $\beta_2 = 0$. β_2/β_1 or β_3/β_1 = — 1/100 — 1/10 — 1 — 10 — 100 — 1,000 — 10,000 — 100,000.

Figure 2 shows free vibration responses of the laminated cylindrical shell for different levels of quadratic and cubic damping coefficients. Fig. 2(a) displays the shell behavior with quadratic damping, and it is possible to observe an effective change since $\beta_2/\beta_1 = 100$. The responses with cubic damping are showed in Fig. 2(b), in which the change is noticeable in a bigger relation, $\beta_3/\beta_1 = 10{,}000$.

Considering these relations for quadratic and cubic damping coefficients, and assuming ζ equal to 0.01 or zero, were obtained the time responses for the laminated shell, totalizing 128 combinations, of which just some will be presented here.

To understand the influence of quadratic damping (considering linear and cubic damping as well), Fig. 3 shows the free vibration behavior for some cases.

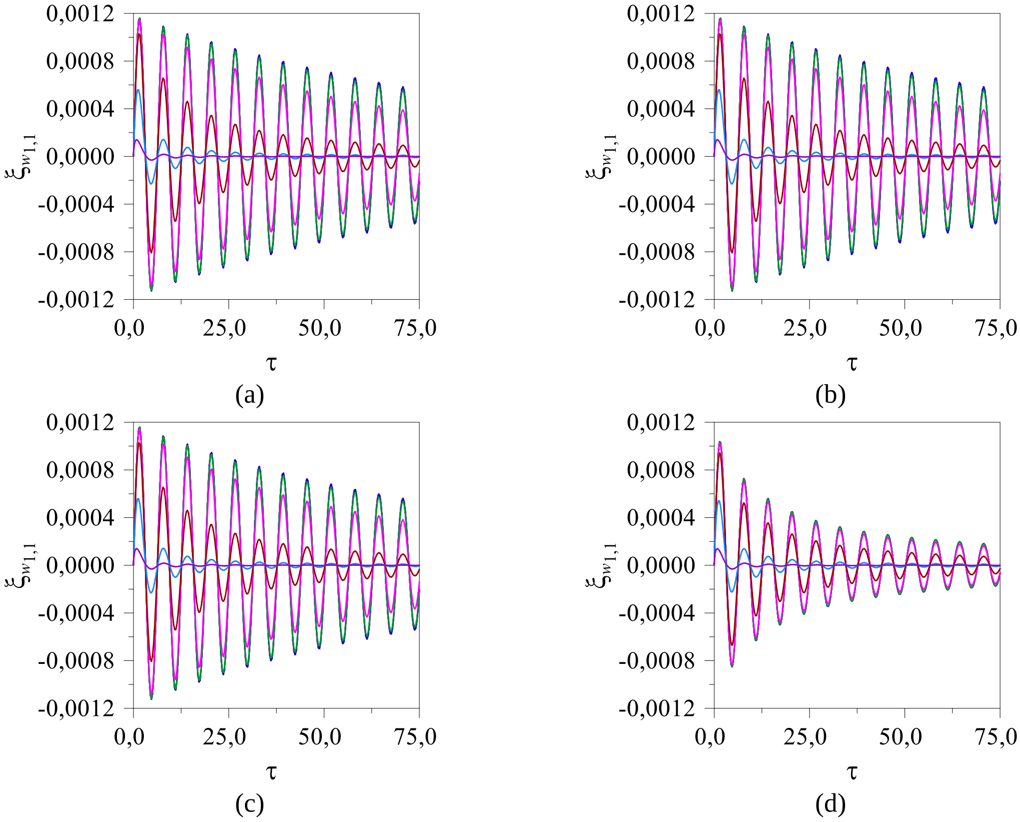

Figure 3. Free vibration response with $\zeta = 0.01$. (a) $\beta_3/\beta_1 = 1/10$, (b) $\beta_3/\beta_1 = 10$, (c) $\beta_3/\beta_1 = 1{,}000$ and (d) $\beta_3/\beta_1 = 100{,}000$. $\beta_2/\beta_1 =$ 0, 1/10, 1, 10, 100, 1,000, 10,000, 100,000.

It can be observed that for small relations the behavior doesn't change, this means that the linear damping is governing the response (as well is seen in the next analysis). From $\beta_2/\beta_1 = 100$ forward it changes, and the quadratic damping has a strong influence, except for the biggest β_2/β_1 where the difference is noticeable for $\beta_2/\beta_1 = 1{,}000$. When ζ is adopted equal to zero, a similar behavior is observed.

In the same way, Fig. 4 displays the free vibration responses with the purpose of understanding the influence of cubic damping coefficients, acting together linear and quadratic damping.

In the first two cases presented (Fig. 3(a) and Fig. 3(b)) the behavior of the shell changes in $\beta_3/\beta_1 = 10{,}000$. Then, it is possible to say that the cubic damping has a small influence for shorter relations, and for bigger values of β_2/β_1 (Fig. 3(c) and Fig. 3(d)) the response is practically the same, independently from β_3/β_1 relation. Similar responses are obtained when ζ is taken null.

Maximum amplitudes of the radial displacement modes were determined from the 128 combinations and then a global sensitivity analysis was performed. It was applied two regression-based methods: Standardized Regression Coefficients (SRC) and Partial Rank Correlation Coefficients (PRCC). The obtained results are shown in Tab. 1 and Tab. 2, respectively.

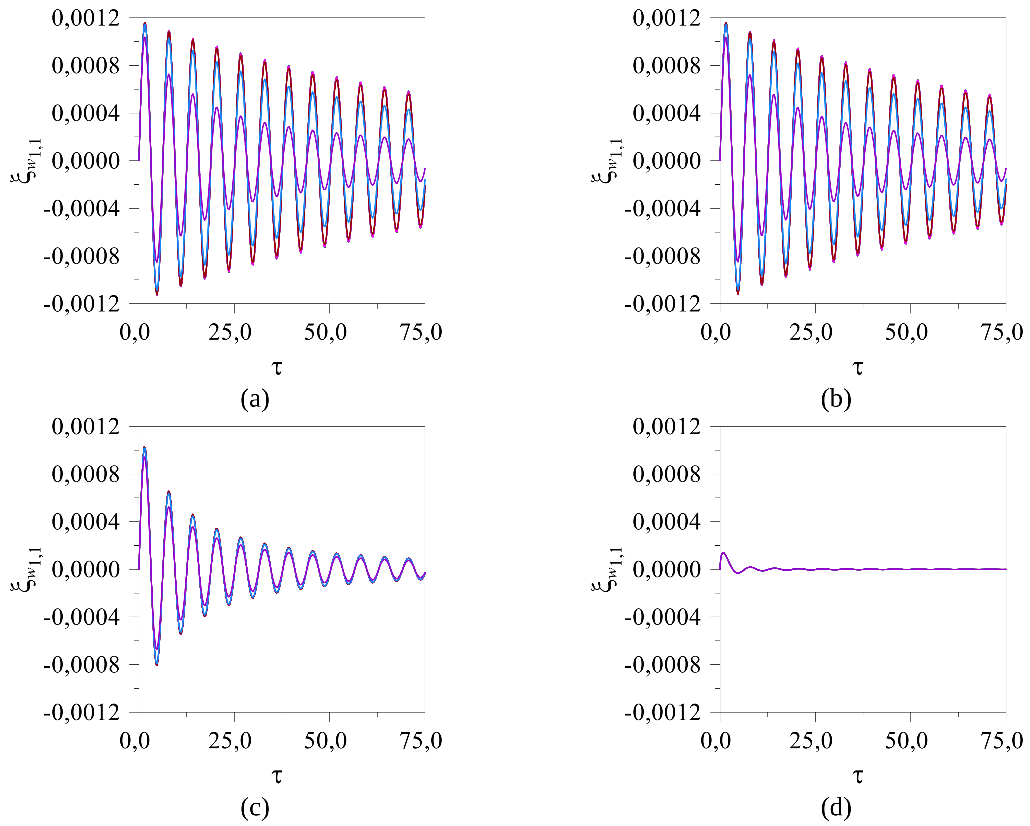

Figure 4. Free vibration response with ζ = 0.01. (a) β_2/β_1 = 1/10, (b) β_2/β_1 = 10, (c) β_2/β_1 = 1,000 and (d) β_2/β_1 = 100,000. β_3/β_1 = — 0 — 1/10 — 1 — 10 — 100 — 1,000 — 10,000 — 100,000.

The SRC technique quantifies the effect of each input variable on the response variables. In other words, for this study, it evaluates the influence of quadratic and cubic damping coefficients on the maximum amplitudes of the radial displacement modes, considering ζ equal to 0.01 or zero.

Table 1. Standardized Regression Coefficients (SRC) for radial displacement modes

	ζ = 0.01			ζ = 0		
Maximum amplitudes	β_2	β_3	Σ^2	β_2	β_3	Σ^2
$\xi_{w1,1}$	-0.88927615	-0.08244842	0.80	-0.88706239	-0.08543799	0.79
$\xi_{w1,1c}$	-0.88929535	-0.08314534	0.80	-0.88750118	-0.08534864	0.79
$\xi_{w0,1}$	-0.9403553	-0.1013922	0.89	-0.9355833	-0.1026132	0.89
$\xi_{w0,3}$	-0.97583013	-0.03534228	0.95	-0.97532443	-0.03699886	0.95

From Tab. 1 it is possible to confirm the strong influence of the quadratic term in the free vibration of the laminated cylindrical shell, far superior to the influence of cubic damping. For driven mode, for example, the influence of quadratic damping is about 89%, while for cubic coefficient is a little more than 8%.

It should be noted that SRC technique is suitable only for linear models; more the sum of the squared coefficients is close to the unity, more the linearity and adjustment of the model is verified [9]. Again, taking as an example the driven mode, the sum is equal to 0.80, which indicates a good approximation.

For nonlinear (but monotonic) models, the PRCC technique is more adequate [9]; it allows verify qualitatively the order of importance of the response variables. The results are shown in Tab. 2 and corroborate that the quadratic damping has the biggest effect on the free vibration behavior of the laminated shell.

Table 2. Partial Rank Correlation Coefficients (PRCC) for radial displacement modes

	ζ = 0.01		ζ = 0	
Maximum amplitudes	β_2	β_3	β_2	β_3
$\xi_{w1,1}$	-0.9530758	-0.7447277	-0.9514657	-0.7456202
$\xi_{w1,1c}$	-0.9530758	-0.7447277	-0.9514657	-0.7456202
$\xi_{w0,1}$	-0.9343142	-0.7348700	-0.9532148	-0.7443547
$\xi_{w0,3}$	-0.9199303	-0.7429571	-0.9250480	-0.7371472

4 Conclusions

The influence of quadratic and cubic damping coefficients on the free vibration of laminated simply-supported cylindrical shell is studied. To model the shell, the Donnell nonlinear shallow shell theory without considering the effect of shear deformation is used and a model with fourteen degrees of freedom describes the displacement field of the shell in axial, circumferential and radial directions.

Rayleigh dissipation function is adopted to describe linear damping, and equivalent expressions are written to represent quadratic and cubic damping coefficients as well. Eight different relations between quadratic and linear damping, even as cubic and linear damping, were selected for ζ = 0.01 and ζ = 0, resulting in 128 combinations. The analysis of time responses for these cases shows the influence of nonlinear damping on free vibration behavior of the laminated shell, which is stronger for the biggest relations between nonlinear and linear damping coefficients.

Also, it is possible to observe that quadratic damping has a bigger influence than cubic one; what is confirmed by the application of Standardized Regression Coefficients (SRC) and Partial Rank Correlation Coefficients (PRCC) techniques.

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