Three-dimensional Dipole BEM formulation for Cohesive Crack Propagation Modelling

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Abstract. This work presents a boundary element method (BEM) formulation for cohesive crack propagation analysis in a 3D approach. BEM is a well-known and remarkable approach in fracture mechanics, providing effective stress concentration modelling in addition to less complex remeshing procedures during crack growth. The fracture effects are captured by using an alternative BEM formulation based on introducing a set of self-equilibrated forces, called a dipole, which describes the cohesive zone. This BEM formulation demonstrates some advantages in comparison to the classical DBEM approach. The DBEM solves the fracture problem with the discretization of both crack surfaces, which leads to six integral equations (three displacements and three tractions) per a couple of points at the crack surface. Alternatively, the dipole approach requires the discretisation of solely one crack surface. Besides, the nonlinear solution scheme corrects the stress components solely at the FPZ, which in the present case are three. Thus, the dipole approach requires solely three integral equations at the FPZ, which is half compared to the DBEM. It leads to faster and more effective performance in terms of computational effort. Some classic examples from the literature are presented in order to validate the 3D Dipole BEM formulation in the light of cohesive crack propagation analysis. Finally, this proposal contributes toward advancing BEM in engineering analyses, especially in nonlinear fracture mechanics.

Keywords: Dipole Boundary Element Method, Three-dimensional crack growth, Nonlinear Fracture Mechanics.

1 Introduction

Realistic and accurate analysis of structural integrity is of utmost importance in engineering. In this context, engineering research becomes essential, and it is interested in developing adequate theories for the prediction of the mechanical behaviour of structures, identifying possible failure and collapse scenarios. The proper description of mechanical fields allows the prediction of material failures, and is of great importance in structural engineering, since its success can save lives, monetary and natural resources. In this way, fracture mechanics arises as an efficient and robust tool for realistic representation of discontinuities, considering that the appearance and the growth of cracks explain the collapse of the material.

In this context, the finite element method has been applied several times in fracture mechanics [1], including its extended version [2]. Alternatively, the boundary element method (BEM) stands out as a powerful technique, especially in cases with stress concentration and extended domains. Therefore, the BEM is an interesting numerical technique to fracture problems, taking account that the remeshing aspects its simplified. One of the most used BEM formulations for fracture mechanics is the dual boundary element method (DBEM), which requires singular and hypersingular integral equations along the crack path, avoiding the division of the solid in subregions. Some remarkable works using the DBEM approach can be mentioned: Xiao et al. [3] and Andrade and Leonel [4]. Other classical formulations using boundary elements have been proposed in the literature for nonlinear fracture mechanics. Among these formulations, it is important to mention the Galerkin BEM [5] and multizone BEM [6].

The extension to 3D approach for crack propagation analysis with BEM formulations has been a topic of great interest in the numerical community. In this regard, it is worth mention some important works: Mi and Aliabadi [7] presented a numerical implementation of the 3D DBEM for elastic fracture mechanics. Xiao and Yue [8] developed a three-dimensional displacement discontinuity method for cracked bodies in layered rocks. Rocha and Leonel [9] presented a BEM formulation of the 3D modelling of concrete structures with the subregion technique and a predefined crack path. Furthermore, the use of 3D BEM formulations for cohesive crack propagation problems still unexplored in the literature.

An alternative formulation for cohesive crack propagations problems its based in the introduction of an initial stress field to represent the fracture process zone (FPZ), the so-called Dipole BEM approach. This formulation is particularly interesting once requires the discretisation of solely one crack surface and three integral equations (related to stress correction) to the representation of the FPZ. In contrast, the dual BEM formulation requires de discretisation of both crack surfaces, which leads to six integral equations (three displacements and three tractions). Moreover, the Dipole BEM formulation leads to a lower computation effort than the classical DBEM formulation. Therefore, Oliveira and Leonel [10] presented a 2D dipole formulation in quase-brittle materials and Almeida et al. [11] expanded to multi-crack propagation analysis.

In this work, the Dipole BEM formulation is extended to the 3D approach considering cohesive crack propagation problems. The Hillerborg [12] model in employed, and the three cohesive laws: Linear, Bilinear and Exponential are used. Further, the hypersingular kernels present in this formulation are regularized with the Guiggiani technique [13] and two examples are presented in order to show the robustness of the 3D Dipole BEM formulation.

2 Dipole BEM formulation

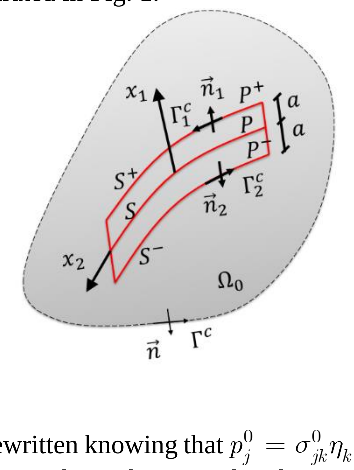
In this section, the alternative BEM formulation for cohesive crack propagation analysis in 3D structures is presented. In this formulation, the nonlinear effects are modelled by imposing an initial stress field, and the influence of the FPZ region in displacement and stress components is computed with the introduction of the dipoles. Further, the domain integral is degenerated along the crack line, which leads to appearance to the variables called dipoles, responsible for correcting the elastic behaviour. For more details of this formulation, see Oliveira and Leonel [10] and Almeida et al. [11].

For sake of completeness, the formulation initially handles the FPZ as a narrow domain positioned in front of the crack tip. This narrow zone has boundary Γ^C, which may be splitted into Γ_1^C and Γ_2^C. Thus, $\Gamma^C = \Gamma_1^C \cup \Gamma_2^C$. Moreover, the thickness of this region is $2a$, which has been assumed as small in comparison with its length. It is worth mentioning that the dipoles appear in the limit of $2a \to 0$, as presented below, Fig. 1.

It is possible to develop the integral representation for displacements by inserting an initial stress field σ_{jk}^0 [14], neglecting the body forces,

$$c_{lk}^i u_k^i + \int_\Gamma p_{lk}^* u_k d\Gamma = \int_\Gamma u_{lk}^* p_k d\Gamma + \int_{\Omega_0} \sigma_{jk}^0 \varepsilon_{ljk}^* d\Omega_0 \tag{1}$$

where u_{lk}^*, p_{lk}^* and ε_{lkj}^* indicate the fundamental solutions for displacements, tractions and strains, respectively [14]. Ω_0 represents the FPZ as illustrated in Fig. 1.



The last term of eq (1) can be rewritten knowing that $p_j^0 = \sigma_{jk}^0 \eta_k$ represents the tractions at the narrow region boundary associated to the FPZ. It is worth emphasizing that the integral terms on eq. (1) have been defined into the local coordinate system, x_ℓ. Nevertheless, the solution of general cohesive crack growth problems requires the evaluation of such terms into the global coordinate system, X_ℓ.

$$\int_{\Omega_0} \sigma^0_{jk}\varepsilon^*_{ljk} d\Omega_0 = \int_{\Gamma^C} 2a \frac{\partial u^*_{lj}}{\partial x_\ell} p^{0\ell}_j d\Gamma^C = \int_{\Gamma^C} 2a \frac{\partial u^*_{lj}}{\partial X_\ell} \overline{p}^{0\ell}_j d\Gamma^C \tag{2}$$

After successive algebraic manipulations and knowing that $q^\ell_j = 2a\overline{p}^{0\ell}_j$, the last term of eq. (2) can be expressed as:

$$\int_{\Omega_0} \sigma^0_{jk}\varepsilon^*_{ljk} d\Omega_0 = \int_{\Gamma^C} 2a \frac{\partial u^*_{lj}}{\partial X_\ell} \overline{p}^{0\ell}_j d\Gamma^C = \int_{\Gamma^C} G^\ell_{lj} q^\ell_j d\Gamma^C \tag{3}$$

where

$$G^\ell_{lj} = \frac{\partial u^*_{lj}}{\partial X_\ell} = \frac{1}{16\pi(1-\upsilon)\mu r^2}\left[-(3-4\upsilon)(r_{,\ell}\delta_{lj} + r_{,j}\delta_{\ell l} + r_{,l}\delta_{\ell j}) - 3r_{,l}r_{,j}r_{,\ell}\right] \tag{4}$$

in which μ indicates the shear modulus, υ represents the Poisson ratio, δ is the Kronecker delta and $r_{,i}$ are the distance derivatives between source and field points.

Therefore, eq. (1) can be rewritten in the following form after the manipulations on the domain term previously presented:

$$c^i_{lk}u^i_k + \int_\Gamma p^*_{lk} u_k d\Gamma = \int_\Gamma u^*_{lk} p_k d\Gamma + \int_{\Gamma^C} G^\ell_{lj} q^\ell_j d\Gamma^C \tag{5}$$

The last equation enables the solution of the boundary value problem accounting for the nonlinear effects at the FPZ. The integral representation for the stress field at internal points, eq. (6), is obtained through the integral displacement representation, eq.(5), and the Hooke's generalized law,

$$\sigma_{lm} = -\int_\Gamma S_{lmj} u_j d\Gamma + \int_\Gamma D_{lmj} p_j d\Gamma + \int_{\Gamma^C} G^{m\ell}_{lj} q^\ell_j d\Gamma^C + g^{m\ell}_{lj}\left[\sigma_{j\ell}(p)\right] \tag{6}$$

in which:

$$G^{m\ell}_{lj} = \frac{1}{8\pi(1-\upsilon)r^3}\begin{Bmatrix} (1-2\upsilon)(\delta_{mj}\delta_{l\ell} + \delta_{lj}\delta_{\ell m} - \delta_{ml}\delta_{\ell j}) + \\ 3(1-2\upsilon)(\delta_{mj}r_{,l}r_{,\ell} + \delta_{lj}r_{,m}r_{,\ell} - \delta_{lm}r_{,j}r_{,\ell}) - \\ 3(\delta_{\ell j}r_{,l}r_{,m} + \delta_{m\ell}r_{,j}r_{,l} + \delta_{\ell l}r_{,m}r_{,j}) - 15r_{,l}r_{,j}r_{,m}r_{,\ell} \end{Bmatrix} \tag{7}$$

and

$$g^{m\ell}_{lj} = \begin{vmatrix} 0 & if\ out\ of\ cohesive\ zone \\ \sigma^0_{im} & if\ on\ the\ cohesive\ zone \end{vmatrix} \tag{8}$$

The crack opening displacement (COD) can be obtained taking account the displacement difference between points symmetrically positioned at $\Gamma^C_1(S^+)$ and $\Gamma^C_2(S^-)$, as illustrated in Fig. 1.

$$\Delta w = \begin{Bmatrix} \Delta w_1 \\ \Delta w_2 \\ \Delta w_3 \end{Bmatrix} = \begin{bmatrix} 1/\mu & 0 & 0 \\ 0 & 1/\mu & 0 \\ 0 & 0 & (1-2\upsilon)/(2(1-\upsilon)\mu) \end{bmatrix} \begin{Bmatrix} q^3_1 \\ q^3_2 \\ q^3_3 \end{Bmatrix} \tag{9}$$

2.1 Algebraic representation

Taking account, the integral representation of displacement and stress field, eqs. (5) and (6), it is important to rewrite these equations in an algebraic form, in order to achieve the nonlinear solution. Therefore, it is convenient to express the problem in a compact form, and eq. (10) can be solved making the necessary changes in matrices H and G.

$$HU = GP + KQ \tag{10}$$

Vector Q contains the approximate dipoles. After the correct rearrangement of the influence matrices H and G (free values in X and prescribed values in F), and knowing that K contains the FPZ representation, the matrix A is obtained, allowing the solution of the algebraic system, as follows,

$$X = M + RQ \tag{11}$$

where,

$$\begin{aligned} M &= A^{-1}F \\ R &= A^{-1}K \end{aligned} \tag{12}$$

As in eq. (10), the stress field, eq. (13), are obtained as following,

$$\sigma = -H'U + G'P + K'Q \tag{13}$$

Matrix K' contains the vector of dipoles Q, H' and G' are the well-known BEM influence matrices. The stress representation is defined as follows,

$$\sigma = N + SQ \tag{14}$$

in which,

$$\begin{aligned} N &= F' - A'M \\ S &= K' - A'R \end{aligned} \tag{15}$$

Regarding the crack propagation, Rankine's criterion is applied to the crack growth stability analysis. Through this rupture criterion, the stress state at the crack tip is compared to the material limit stress [11]. Internal points are placed near the crack tip in a circumferential arrangement, and the choice of internal points is made based on the desired accuracy. Regarding the crack propagation direction, the theory of maximum circumferential stress is used, and the crack is considered to propagate perpendicularly to the maximum circumferential tensile stress, as stated in Almeida et al. [11].

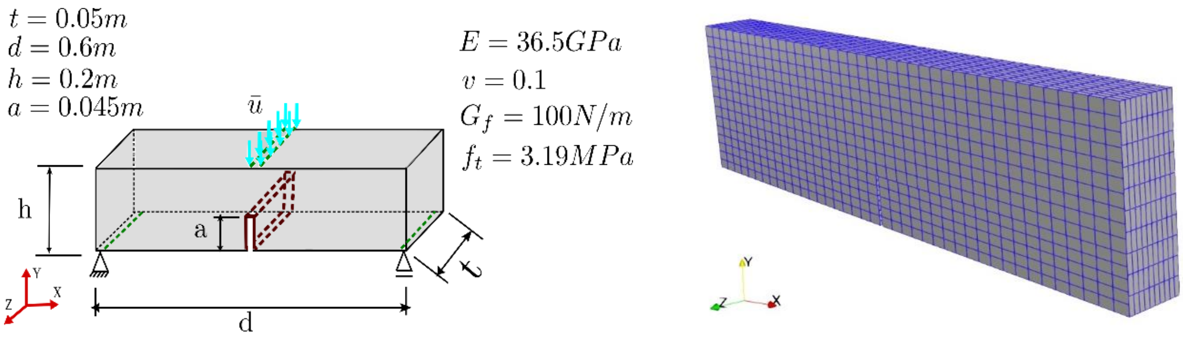
It is important to mention that the strong singularity present in the kernels of eq.(14) require a special regularization technique for integrating the hyper singular crack element. The regularization technique utilised in this work was presented in Guiggiani [13]. Furthermore, the crack elements are always discretised with discontinuous linear quadrilateral elements, taking account the hyper singularity present in this formulation.

3 Applications

In this section, two examples were chosen to illustrate the accuracy and robustness of the proposed 3D formulation. In both examples, the three cohesive laws were employed: Linear, Exponential and Bilinear. The nonlinear system solution is solved with a tolerance of 10^{-2} , as a function of non-equilibrated stress values. Experimental and numerical results are provided from the literature.

3.1 Example 1: Three-point bending test

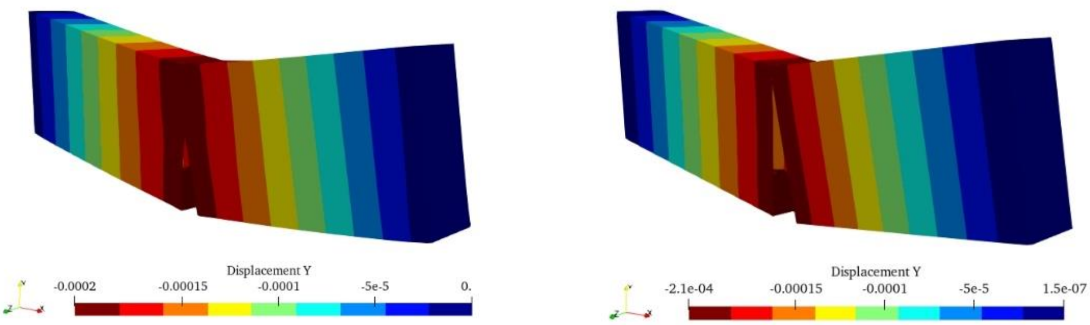
In this example, a concrete specimen subjected to a three-point bending test is presented. The analytical [15] and numerical results [16], with the standard finite element method, are available in the literature. The boundary mesh consists of about 3441 collocation points and 2900 quadrilateral linear boundary elements, Fig. 2(b). The dimensions and materials properties are presented in Fig. 2(a).



(a) (b)

Figure 2. Three-point bending specimen (a) Boundary mesh discretisation (b)

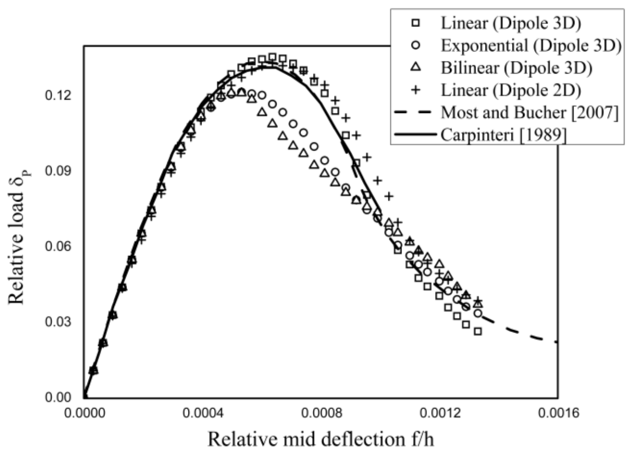
In Figure 3, the crack propagation analysis is presented for two diferent steps. Further, it is observed that the specimen is close to collapse, however, there is no separation of the structure into two parts.



(a) (b)

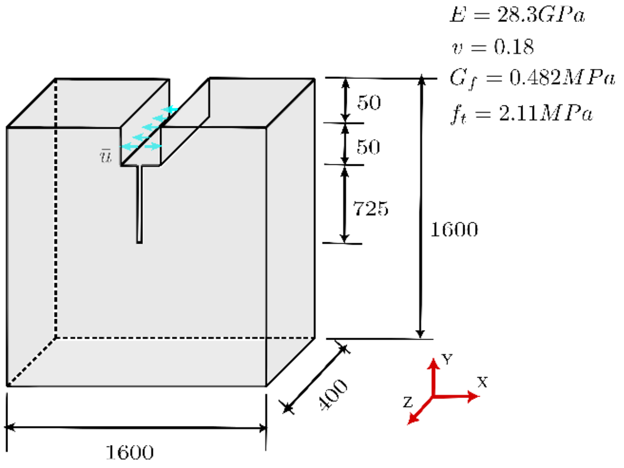
Figure 3. Crack path in colour scale (m): (a) Initial; (b) Final

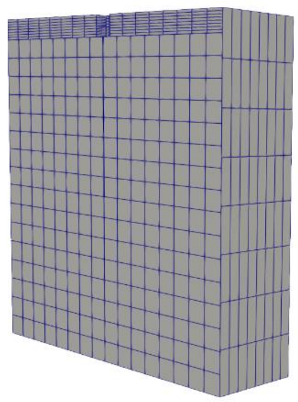
In Figure 4, the relative load, $\delta_P = P / \; t \cdot h \cdot f_t$, *versus* relative mid deflection (given by the quotient of the mid deflection, f , and the beam height, h) curve is presented for the 3D proposed formulation. The parameter P is the equivalent vertical force. Excellent agreement with the analytical result [15] is observed. Further, the linear cohesive law presented better results when compared with the others constitutive laws. Moreover, equivalent results with the 2D Dipole formulation can also be mentioned.



3.2 Example 2: Concrete specimen in Mode I (Wedge-Splitting Test)

In example 2, a concrete specimen with experimental results [17], is analysed by using the 3D Dipole BEM formulation. The dimensions and materials properties are presented in Fig. 5(a). The boundary mesh consists of about 2754 collocation points and 2176 quadrilateral linear boundary elements, Fig. 5(b).

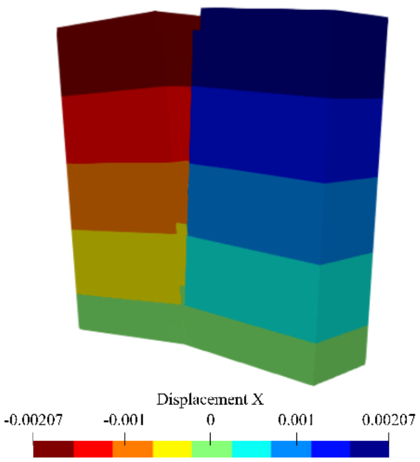


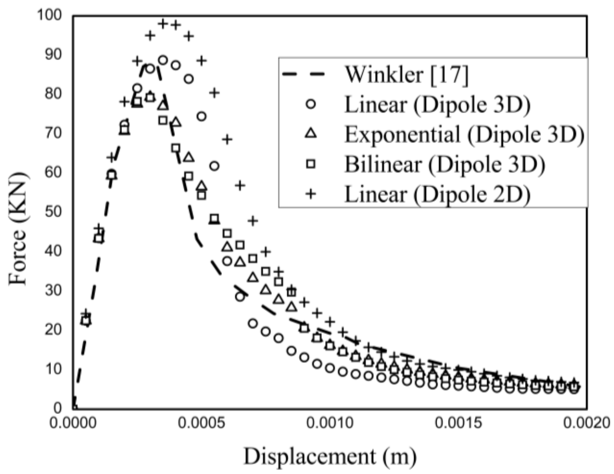


(a) (b)

Figure 5. Wedge-Splitting specimen (a) Boundary mesh discretisation (b)

In Figure 6(a), the horizontal displacement field (m) is presented in colour maps. There is a tendency to separate the solid in two independent parts, with the evolution of the imposed displacement. In Figure 6(b), it can be observed that the proposed 3D formulation was capable to represent the nonlinear behaviour and presents satisfactory responses in terms of force *versus* displacement curve. Futher, the exponential law presented a noticeable agreement with the experimental curve in the post peak load, and it is worth mentioning that the results obtained by de 3D Dipole BEM formulation are equivalent with the 2D Dipole formulation.





(a) (b)

Figure 6. Horizontal crack path in colour scale (m), Force *versus* displacement curve

4 Conclusions

In this work, a tridimensional nonlinear BEM formulation for cohesive crack growth modelling was proposed. Such formulation is based on dipoles of stresses, which lead to an alternative approach to the classical dual BEM. The 3D Dipole BEM formulation proved to be a powerful technique to the modelling of cohesive crack growth, and its robustness and efficiency can be demonstrated in the light of the examples that were presented in this work. In addition, the 3D Dipole approach proved to be more accurate that the 2D Dipole BEM formulation.

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