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On crack simulation by mixed dimensional coupling in GFEM Global-Local

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Abstract. The Generalized Finite Element Method (GFEM) is a numerical method established as an alternative to Finite Element Method (FEM). Considered as an instance of the Partition of Unity Method (PUM), the GFEM uses enrichment functions that, multiplied by the Partition of Unity (PU) functions, expand the space of the solution problem. These enrichment functions could be chosen according to the problem analyzed or numerically obtained from the results of the analysis of a local problem, in the GFEM global-local strategy. Extending the field of application of this method, the global-local Generalized Finite Element Method (GFEM) is used here to solve mixed dimensional structural problems. Combining mixed-dimensional elements and a multi-scale analysis can be highly effective to capture the local structure features without overburdening the global analysis of the problem. An iterative procedure, which balances the forces between the two multi-dimensional models, was automated and combined with the global-local analysis of GFEM. This new procedure incorporated into a computational system made possible the simulation of quasi-static crack propagation. In the numerical example a small scale plane stress problem, where the crack propagates, is coupled with a large-scale model described by Timoshenko beam elements.

Keywords: Generalized Finite Element Method, Global-Local Enrichment, Constraint Equations, Mixed-dimensional Coupling, Multi-scale modeling

1 Introduction

A wide variety of engineering problems can be described by mathematical models in which state variables suffer large variations over small domains [1]. This local phenomena could be problems involving stress concentrations, crack initiation and propagation.

The need to use a large number of elements that allow the discretization of these localized phenomena, causes a excessive computational cost. In order to provide good solutions without using a large number of small elements, Noor [2] proposed a two scale global-local analysis. In this strategy, the solution of a global problem with a coarse discretization is used to provide the boundary data for a local analysis solved with a fine discretization. Instead of performing two separate analysis, coupling techniques can also be employed, allowing a concurrent multi-scale modelling [3–7].

The challenging issue in the multi-scale FE modeling is therefore how to guarantee the rationality of the coupling method so that it can achieve both displacement continuity and stress equilibrium in the region around the interface between the different types of elements [7]. There are different coupling methods that insure this continuity, and one of them is the multipoint constraint (MPC) method, described in Felippa [8].

In this paper, the Generalized Finite Element Method (GFEM) approach (Strouboulis et al. [9] and Duarte et al. [10]) is used in a mixed dimensional global-local analysis using three different interconnected models. Firstly, a multidimensional global model is considered, with two problems discretized by different finite element formulations in different dimension levels. To create a coupling method that guarantee the continuity in displacement and stress balance at the interface between these two problems, the iterative procedure introduced by Wang et al. [7] is applied. A second model, named nodal force model, is created. Its solution defines a restriction equation matrix that relates the different types of degrees of freedom at the interface between the two problems of the first model. The third model is the local problem associated with the higher dimensional level problem of the first model and it is discretized by a very fine mesh. It embraces the region where one or more local features of interest are present. The first and the third models are related by the strategy proposed by Duarte and Kim [11] in the

Global-local GFEM (GFEM^{gl}). The numerical solution obtained from the third problem is used as enrichment function to improve the approximation of the first problem.

Aiming to investigate and validate this method in a crack propagation analysis, a simple problem is considered. A beam is globally represented by Timoshenko elements. The local behavior is reproduced in a two dimensional mesh considering plane stress state, where the crack propagation is evaluated. The results are compared to standard analysis with GFEM.

2 Theoretical Foundation

2.1 Generalized finite element method (GFEM)

In GFEM [9, 10] the approximate function can be improved by the extrinsic enrichment of standard partition of unity functions (PU), such as the FEM shape functions $N_j(\mathbf{x})$. Local functions $L_{ji}(\mathbf{x})$ multiply the PU resulting in the approximate function $\phi_{ji}(\mathbf{x})$ given by:

$$\{\phi_{ji}\}_{i=1}^{q_j} = N_j(\mathbf{x}) \times \{L_{ji}(\mathbf{x})\}_{i=1}^{q_j}. \tag{1}$$

The local approximation functions $L_{ji}(\mathbf{x})$ can be polynomial functions previously established, or special functions chosen from an a priori knowledge of the behaviour of the problem.

Using the concept of solution decomposition into two scales of analysis, Global Local Generalized Finite Element Method, GFEM^{gl}, provides a framework to enrich the global problem solution space with functions numerically constructed from the solution of a local boundary value problem [12].

The numerical solution obtained in the local problem ($\tilde{\boldsymbol{u}}_L$) is used in eq. 1 as the local approximation function $L_{ji}(\mathbf{x})$ to build the approximate solution of the glocal-local enriched problem.

Crack simulation by GFEM

The enrichment functions used in this work to describe the crack are a combination of the Heaviside functions and asymptotic functions proposed by Szabo and Babuska [13].

The Heaviside function corresponds to a step-type enrichment [1, 14, 15], defined according to the eq. 2.

$$H(x) = \begin{cases} 1 & \forall \quad \xi > 0 \\ 0, & \forall \quad \xi < 0 \end{cases} \tag{2}$$

where ξ represents the position with respect to the discontinuity assumed at $\xi = 0$.

The asymptotic functions that describe the singularity in the stress field at the crack tip [1] can be described through the equations:

$$u_x^{(1)} = \frac{1}{2G} r^{\lambda^{(1)}} [(\kappa - Q^{(1)}(\lambda^{(1)} + 1)) cos\lambda^{(1)}\theta - \lambda^{(1)} cos(\lambda^{(1)} - 2)\theta] \tag{3}$$

$$u_y^{(1)} = \frac{1}{2G} r^{\lambda^{(1)}} [(\kappa + Q^{(1)}(\lambda^{(1)} + 1)) sin\lambda^{(1)}\theta + \lambda^{(1)} sin(\lambda^{(1)} - 2)\theta] \tag{4}$$

2.2 Mixed-dimensional finite element coupling

The multi-scale FE modeling needs FE coupling methods to combine mixed-dimensional finite elements, such as beam-to-shell and plate-to-solid, in a single structural model [7]. Thus, choosing the most appropriate coupling method is essential to ensure both the continuity of displacements and the balance of stress at the interface between the two distinct elements.

When linking the nodes of the different elements that compose an interface, a constraint equation is defined containing the displacements of this nodes, establishing an equation of the MPC (multiple point constraint) type. The numerical coupling method proposed by Wang et al. [7] uses the principle of virtual works to obtain, iteratively, at the interface, both the force restraint equations and the displacement restraint equations.

A substructure from the interface of the plate elements is used to create the nodal force model (Fig. 1). At each end of the model a node is created that represents the beam at the interface.

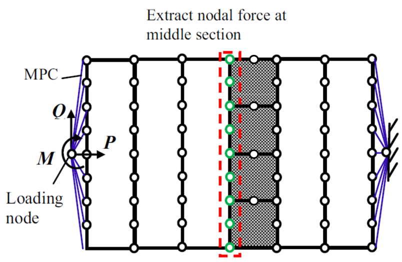

Figure 1. Nodal Force Model, adapted from Wang et al. [7]

An iterative process is performed in the nodal forces model:

- The rigid connection between the different nodes is used to execute the first analysis in the process.
- The result from the analysis is used to create a coefficient matrix.
- The rigid connection is replaced by a deformable connection
- From the second analysis, the nodal force model is performed using the deformable multipoint constraint equation
- The procedure continues until there is a convergence, under a chosen tolerance, for the values of these coefficients
- The matrix of the displacement constraint equation obtained in the last iteration is then transferred to the main plate-beam model

3 Computational Aspects

The objective of this work is to analyse the performance in the simulation of a multidimensional problem in the context of Linear Elastic Fracture Mechanics. In this strategy, a numerical procedure was formulated that combines the GFEM with Global-Local enrichment with mixed-dimensional coupling techniques, enabling the study of the crack propagation in structures. The process of this strategy in a crack propagation model is illustrated in Fig. 2 and described below.

1. The discretization of the global model is defined. A coarse mesh is adopted, without the representation of the initial notch.
2. The substructure is extracted from the plate elements in the interface and used to create the nodal force model.
3. The iterative process is performed in the nodal forces model. The coefficient matrix is obtained from the displacement constraint equations.
4. The coefficient matrix is transferred to the main plate-beam model and the initial global mixed-dimensional model is solved.
5. The global elements that will create the local model are chosen from the points that define the initial crack, following the procedure proposed by Fonseca et al. [16].
6. The local model is then built from the mesh refinement of the elements selected in the previous step and solved.
7. The description of the crack is taken to the global problem through the enrichment of the nodes highlighted in blue in Fig. 2. The same coefficient matrix obtained in the step 3 is used to solve the enriched global problem. To improve the quality of the local problem, the global-local cycles are executed (see [1])
8. The steps 5, 6 and 7 are repeated until the last crack propagation step.

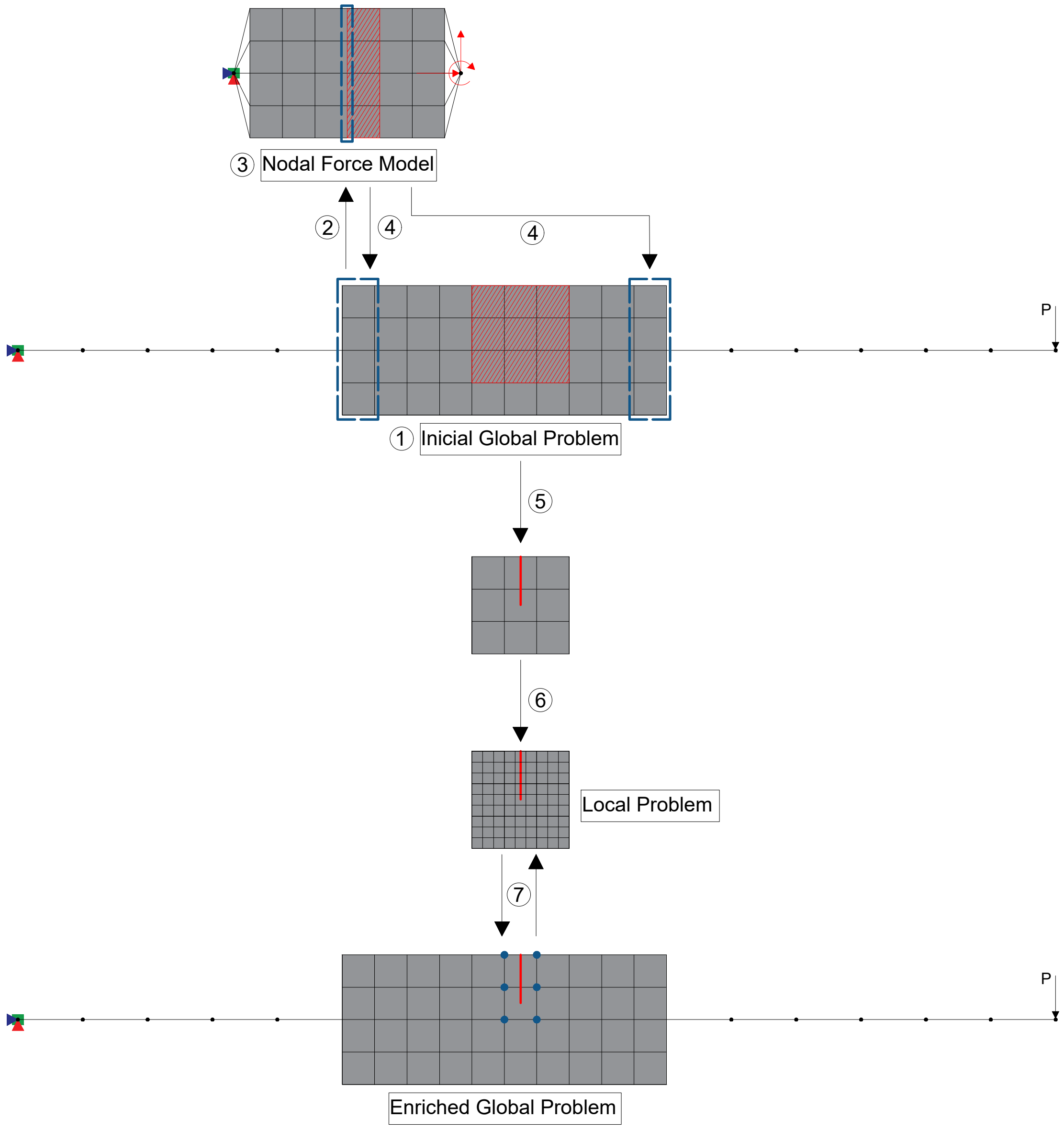

Figure 2. Mechanism for solving the crack simulation by mixed dimensional coupling in GFEM Global-Local

4 Numerical Results

In Fig. 3, the structural problem of a cantilever beam with an edge crack is presented.

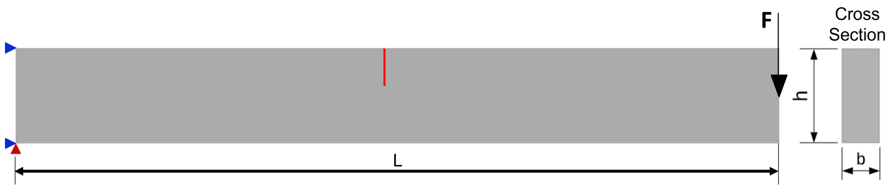

Figure 3. Cantilever beam

The problem has length $L = 320mm$ and cross section with $h = 40mm$ and $b = 8mm$. An isotropic linear elastic material was employed with elastic modulus of $E = 2 \times 10^5 N/mm^2$ and Poisson ratio $\upsilon = 0.3$. The right end of the beam is loaded with a force $F = 12,000N$. Except from the presence of the crack, this problem is similar to the one studied by McCune et al. [17] and Wang et al. [7] for the evaluation of the coupling between beam and plate models and evaluated in Gomes and Barros [18].

Based on the model of the cantilever beam, two numerical simulation are presented in order to show that the strategy proposed here is able to simulate the crack propagation from an initial notch in a problem that is globally represented by Timoshenko beam elements.

The model A, considered as the reference solution. A mesh of 3200 Q4 elements under plane stress condition was used in a GFEM analysis. A quadratic approximate solution was obtained by the enrichment of polynomial functions of the first degree in all nodes.

The model B, the beam problem is discretized in a multidimensional model subdivided into three sections. The first $100mm$ and the last $120mm$ were discretized with a total of 44 linear Timoshenko beam elements. The central region with length of $100mm$ was represented by a mesh of linear Q4 elements, equivalent to the one adopted in Model A. In this problem, the behavior of the model was analyzed with the inclusion of a initial notch in the center of the section with the plate model.(Fig. 4).

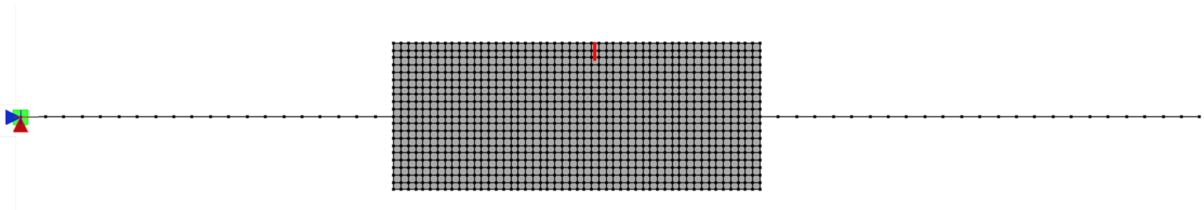

Figure 4. Mixed-dimensional model. The initial notch is represented in red.

The Fig. 5 presents the crack trajectory for both models.

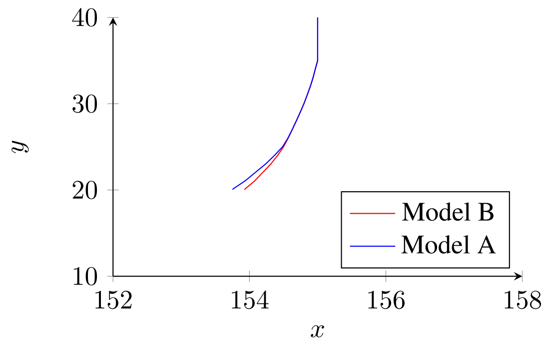

Figure 5. Crack trajectory of the two models

The Fig. 6 and Fig. 7 presents the ratio between the results of stress intensity factor KI and the strain energy, respectively, from the model A and Model B from each step of the crack propagation analysis.

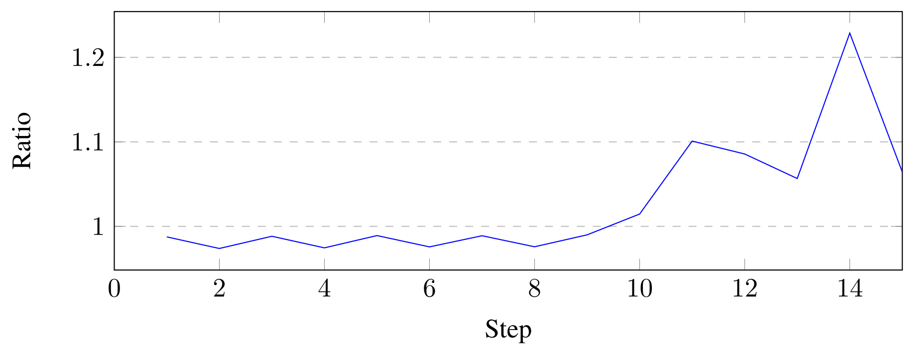

Figure 6. Ratio of the KI parameter

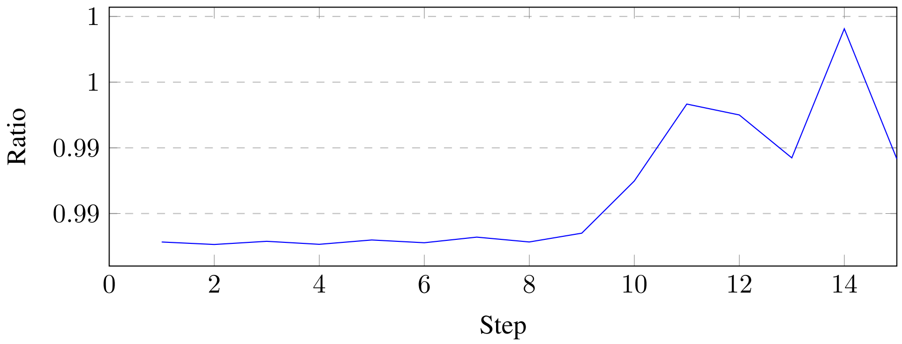

Figure 7. Ratio of the strain energy

It can be seen from the results that the multi scale problem could capture the behavior of the crack propagation along the steps of the analysis, delivering results closer to the reference model A.

5 Conclusions

It is known that it is possible to use elements with different discretization scales in the same analysis model, due to the existing coupling methods. In this way, a local phenomenon need not be described, necessarily, with the same type of element as the global model, if only some characteristics of this local problem are relevant. Thus, the global analysis can be performed with a different mathematical model than the local model analysis.

Analyzes that use the Global-Local strategy allow the refining of the finite element mesh only in the regions of the structure where local phenomena can occur, without impacting the mesh of the global problem. This is a widely used analysis to facilitate the study of complex structures, as it allows reducing the number of degrees of freedom of the model without losing the quality of the solution, reducing the time of assembling the model mesh and saving on the processing time of analysis.

Given the consolidation of these two approaches, both multidimensional modeling and GFEM^{gl}, for analysis of complex structures, in this work the new technique developed to perform analyses via GFEM^{gl}, using the multidimensional modeling was used to explore crack propagation analysis in bi dimensional models.

From the results obtained, it can be concluded that this new technique was able to simulate the crack propagation in a multidimensional model, where the global behavior is simulated by the Timoshenko beam formulation.

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