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# Efficacy of an Adaptive Integration Scheme on the Numerical Performance of DIBEM applied to the Solution of Compressible Diffusive-Advective Problems

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**Abstract.**
Advection-diffusion models can adequately describe several industrial applications in relevant areas such as heat and mass transfer, fluid flow, metallurgy, pollutant dispersion among a wide spectrum of engineering problems. This class of problems presents challenging numerical aspects to the boundary element method (BEM), in special for formulations that employ radial basis functions to approximate the advective domain integral as occurs in the Direct Interpolation technique (DIBEM). Furthermore, the representation of variable velocity fields and the reproduction of compressibility effects in low to moderate Peclet flows also require a more robustness numerical model. For definition, BEM formulations generates singular and quasi-singular integrals that demand an adequate treatment. Specifically, the application of DIBEM requires a greater number of interpolation poles in the domain for a better accuracy. As the internal points are also source points, more refined meshes require the use of adaptive schemes to handle the quasi-singular integrals that arise from locating the source points closer to the boundary. In the present article, the performance of the Telles's self-adaptive integration scheme is compared to the classical Gaussian quadrature, in a two-dimensional diffusive-advective application with variable velocity. DIBEM results are compared with Dual Reciprocity technique (DRBEM) and the available analytical solution. These preliminar results indicate that the use of the adaptive scheme provides more significant improvements in accuracy for DRBEM when compared to DIBEM.

**Keywords:** Boundary Element Method, Direct Interpolation Technique, Diffusive-Advective Model, Self-Adaptative Integration Scheme.

## 1 Introduction

In the last three decades, the advance of the Boundary Element Method (BEM) tools in the treatment of advective-diffusive models is remarkable. In the context of BEM, stands out the classic formulation, it uses the fundamental solution related to the advective-diffusive model, which can be found in the literature of [1], to generate the inverse integral formulation of the problem. This formulation, which is well illustrated in the work of [2], is capable to describe physical situations with advection dominance without present stability problems, however, its application is restricted to uniform velocity fields. Despite the mathematical elegance and attractive proposal of the fundamental solutions of the boundary element method, the complexity of the differential operator often makes it very difficult to employ the classic formulation of the method.

The proposal of the Dual Reciprocity technique (DRBEM) presented by the work of Nardini and Brebbia [3], gives flexibility to the boundary element method, since it allows the use of a simpler fundamental solution, together with the convenient approximation of domain integrals via radial base functions [4]. This technique, which is widely spread by BEM researchers, it has been tested in a systematic way on several scalar field problems in different papers such as those of [5] and [6]. Concerning advective-diffusive models, the work of [7] illustrates the limitation of DRBEM dealing with low to moderate Peclet numbers, in uniform velocity field situations, while in another work, the same author states the flexibility of the technique in the face of variable velocity fields [8].

The accuracy of an integral formulation, such as the BEM, it necessarily passes through a precise determination of the matrices coefficients that constitutes the discrete linear system. In order to better approximate the singular and quasi-singular integrals that appear in the boundary element codes, the work of [9] stands out, which proposes a method based on coordinate transforming, that is capable of handling quasi-singular and also singular integrals, with good precision for three dimensions and for curved elements. Telles [10] proposes an ingenious transformation of coordinates based on a cubic polynomial, where the Gaussian points are shifted depending on the position of the singularity, which allows a more precise assessment of the integrals of interest. The previous author [11] expand the line of research in the previous article by coupling an adaptive mechanism to its integration proposal, which makes it even more robust and therefore useful. The work of [12] proves that Telles's proposal shows adherence with DRBEM, and that the results of the technique are significantly improved by using this integration scheme, in several scalar field problems.

Loeffler, Cruz and Bulcão [13] proposes a formulation of Boundary Elements called the Direct Interpolation Method (DIBEM), where the entire kernel of the domain integrals are approximated by radial basis functions, thus being a proposal structurally similar to the DRBEM. In this paper [13], DIBEM was applied in two-dimensional problems with domain action and notes a superior performance of this new proposal in relation to the DRBEM technique. In a logical and systemic sequence, DIBEM technique [14] is successfully applied to two-dimensional Helmholtz problems, obtaining satisfactory accuracy. In [15] Loeffler and Mansur propose a regularization process that deals with the singularity issue and simplifies data entry in DIBEM codes, which is now part of the standard configuration of the technique. Regarding advective-diffuse problems, [16] applies DIBEM using regularization procedure to stationary two-dimensional problems with uniform velocity, in comparison with DRBEM, and notes that this formulation maintains stability at moderate Peclet numbers for the tested cases. In this sort of problems there is still a demand to test the DIBEM in convection problems with variable velocity fields and for a more general direct comparison with DRBEM approach.

This article aims to evaluate the impact of Telles' integration scheme on the accuracy of DIBEM formulation, in advective-diffusive problems. Therefore, the results of the implementations with this proposal are compared with those generated by the classic Gaussian quadrature and DRBEM technique, and all errors are quantified via analytical solutions. The motivation of this work is based on the well-known accuracy improvement that Telles' integration scheme promotes in the formulation of Dual Reciprocity (DRBEM), therefore, it is natural to start to investigate this possible optimization on the DIBEM technique. The initial tests were carried out in uniform velocity field situations [17] and the demand to determine Telles's quadrature adherence in cases of variable velocity is still relevant to the DIBEM formulation advances.

## 2 Self-Adaptative Integration

One of the most used techniques in the numerical approximation of these integrals is the Gauss-Legendre quadrature, whose more formal definition can be found in [18]. The most basic unidimensional concept of this tool is based on an approximation in a $[-1, 1]$ domain, where the integral of a generic $f(x)$ function in this range can be approximated by a series combination of products between the image of the function at specific points of the cartesian, called Gaussian points $x_n$, weighted by scalar values $\omega_n$, called Gaussian weights, according to the algebra brought by Eq. (1) below.

$$I = \int_{-1}^{1} f(x)dx = \sum_{n=1}^{GPN} w_n f(x_n) \tag{1}$$

In order to explain the Telles' adaptative quadrature proposal, which is a competitive option to the classic gauss quadrature, then consider an integration of a function $f(\eta)$ in a range of $[-1, 1]$, represented by Eq. (2).

$$I = \int_{-1}^{1} f(\eta)d\eta \tag{2}$$

First, in his article [10] proposes a transformation of variables based on a third degree polynomial for calculating singular integrals, whose independent variable is $\gamma$, and shown in its generic form in Eq. (3), whose singularity point is at $\overline{\eta}$.

$$\eta(\gamma) = a\gamma^3 + b\gamma^2 + c\gamma + d \tag{3}$$

In the equation Eq. (3), the argument $\gamma$ corresponds to the alterede Gauss points, after the transformation and $\eta$ represents the weights associated with the integration points $\gamma$. In order to determine the coefficients $a, b, c, d$ of the polynomial, which are adjusted according to the position of the singularity, Telles proposed the imposition of the following boundary conditions:

$$\eta(1)=1 \qquad \eta(-1)=-1 \qquad \frac{d\eta}{d\gamma}_{\gamma=\overline{\gamma}}=0 \qquad \frac{d^2\eta}{d\gamma^2}_{\gamma=\overline{\gamma}}=0 \tag{4}$$

These boundary conditions shown by the set of Eq. (4) inidicates, in the first two, that the interval of the transformed function does not change and remains at $[-1,1]$, which coincides with use of isoparametric elements in generalized coordinates in BEM. As well explained in the work of [12], the third condition implies that the Jacobian of the proposed transformation, it should preferably reduce the order of the singularity at $\overline{\eta}$. The fourth condition imposes a minimum point for the Jacobian of the transformation $J(\gamma)$. In other words it means that the Jacobian has its minimum value coinciding with the point where the function is singular, therefore, it presents more severe gradients. The Jacobian expression, as the transformation that is performed with it, is shown below.

$$I=\int_{-1}^{1} f(\eta(\gamma))J(\gamma)d\gamma \qquad J(\gamma)=\frac{d\eta}{d\gamma}=\frac{3(\gamma-\overline{\gamma})^2}{1+3\overline{\gamma}^2}=3a\gamma^2+2b\gamma+c \tag{5}$$

The pure and simple use of the boundary conditions of Eq. (4) for the determination of the coefficients of the third degree polynomial was made, and is available in [10], however, these coefficient values are restricted if, in the context of boundary elements, the coincidence of the source and field points is located in the element's range, that is, the singularity occurs in over the boundary element, characterizing a singular integral.

Telles' second relevant contribution was to parameterize the entire mathematical model of the quadrature as a function of the distance $R_{min}$. This adaptation is necessary for the case of quasi-singular integrals of any intensity, depending on the distance between the source point, which in this case can be imposed to any generic position, and the interpolant one. The sequence for this so-called self-adaptive quadrature begins by determining the shortest distance between the source point $\xi$ and the element, called $R_{min}$ and thereby calculating the value of the parameter $D$ with the equation Eq. (6).

$$D=\frac{2R_{min}}{L} \tag{6}$$

In Eq.(6), the parameter $L$ is taken as the distance between the extreme nodes of the element. This distance is taken between the nodes $\eta=-1$ and $\eta=1$. With $D$ it is possible to determine the adaptation parameter $\overline{r}$ according to the intervals presented by [10], and following the ranges of the equation (7).

$$\begin{aligned} \overline{r}&=0 && 0.00\le D<0.05\\ \overline{r}&=0.85+0.24\ln(D) && 0.05\le D<1.30\\ \overline{r}&=0.893+0.0832\ln(D) && 1.30\le D<3.618\\ \overline{r}&=1 && D\ge 3.618 \end{aligned} \tag{7}$$

Once the adaptive parameter $\overline{r}$ is known, it is possible to calculate the auxiliary variables $p$ and $q$ represented by the equations (8) and (9), respectively.

$$p=\frac{1}{3(1+2\overline{r})^2}\left[4\overline{r}\,(1-\overline{r})+3\left(1-\overline{\eta}^2\right)\right] \tag{8}$$

$$q=\frac{1}{2\,(1+2\overline{r})}\left[\left(\overline{\eta}\,(3-2\overline{r})-\frac{2\overline{\eta}^3}{1+2\overline{r}}\right)\frac{1}{1+2\overline{r}}-\overline{\eta}\right] \tag{9}$$

The parameters $p$ and $q$, such as the position $\overline{\eta}$ of the singularity are used by the Eq. (10), to determine $\overline{\gamma}$, which is the position of the singularity in terms of the transformed variable. This parameter is very relevant, since the coordinate transformation performed by the Jacobian depends on the distance from each point of Gauss to the singularity, as already shown by the equation (5).

$$\overline{\gamma}=\sqrt[3]{\left[-q+\sqrt{q^2+p^3}\right]}+\sqrt[3]{\left[-q-\sqrt{q^2+p^3}\right]}+\frac{\overline{\eta}}{1+2\overline{r}} \tag{10}$$

Finally, knowing the the adaptation parameter $\overline{r}$, the coefficients $a,b,c,d$ initially deducted in [10] for singular integrations only, can be generalized for any position of the source point $\xi$, inside or outside the element, by Eq. (11).

$$a=\frac{(1-\overline{r})}{Q} \qquad b=-\frac{3\overline{\gamma}\,(1-\overline{r})}{Q} \qquad c=\frac{\left(\overline{r}+3\overline{\gamma}^2\right)}{Q} \qquad d=-b \tag{11}$$

## 3 Direct Interpolation Tecnhique

The diffusion-advection equation is widely used to model and describe physical phenomena where there is the transport of a certain physical quantity, such as mass or energy, together with the diffusion process, in a given volume of control. Considering a permanent regime, in the absence of sources or sinks, with constant and unitary diffusion coefficient and uniform velocity field, in indicial notation we have:

$$u_{,ii} = v_i u_{,i}. \tag{12}$$

In Eq. (12), the left side represents the diffusive effects (DS), while the right side represents the accounting of the advection, where the variable velocity field $v_i$ carries the scalar field $u$. The treatment via BEM on the diffusive side is already widely known in the literature [19] and its inverse integral formulation corresponds to the Eq. (13) below.

$$DS = c(\xi)u(\xi) + \int_\Gamma u(X)q^*(\xi;X)d\Gamma - \int_\Gamma q(X)u^*(\xi;X)d\Gamma. \tag{13}$$

The domain integral, which quantifies the advective part (AS) has, a priori, a more challenging treatment, whose mathematical structure is shown below. Each portion of the function in the integrands is kept explicit in order to promote a deeper consistency in exposition.

$$AS = \int_\Omega v_i(X)u_{,i}(X)u^*(\xi;X)d\Omega. \tag{14}$$

The application of integration by parts in Eq. (14), in addition to the ensuing use of the Divergence Theorem [18] in order to move the integrals to the boundary $\Gamma(X)$ lead to the following equation.

$$AS = \int_\Gamma n_i(X)v_i(X)u(X)u^*(\xi;X)d\Gamma - \int_\Omega v_i(X)u^*_{,i}(\xi;X)u(X)d\Omega - \int_\Omega v_{i,i}(X)u^*(\xi;X)u(X)d\Omega. \tag{15}$$

In Eq.(15), the first integral is already written in terms of the boundary $\Gamma(X)$, there still is, however, a domain integral where the velocity fields $v_i$ and another one, which is also written in $\Omega(X)$ whose domain is composed by the divergent of the velocity fields $v_{i,i}$ that is connected to the compressibility effects of the flow [20]. At this point, similarly to the procedures used in an incompressible flow formulation whose details can be examined in [21], a regularization process is conducted in the second integral of Eq. (15), which is analogous to Hadamard's proposal [22]. This procedure already integrates the DIBEM's standard formulation and can also be viewed in previously stablished works [15].

$$\begin{aligned} AS = \int_\Gamma v_i(X)n_i(X)u(X)u^*(\xi;X)d\Gamma + \int_\Omega v_i(X)u^*_{,i}(\xi;X)[u(\xi) - u(X)]d\Omega \\ + \int_\Omega v_{i,i}(X)u^*(\xi;X)[u(\xi) - u(X)]d\Omega - \int_\Omega v_i(X)u^*_{,i}(\xi;X)u(\xi)d\Omega - \int_\Omega v_{i,i}u^*(\xi;X)u(\xi)d\Omega. \end{aligned} \tag{16}$$

Now, we have two surplus domain integrals from the regularization process in Eq. 16. It is possible to rewrite the last integral of Eq. 16, using the product rule as follows:

$$\int_\Omega v_{i,i}u^*(\xi;X)u(\xi)d\Omega = \int_\Omega (v_i u^*(\xi;X)u(\xi))_{,i}d\Omega - \int_\Omega v_i u^*_{,i}u(\xi)d\Omega. \tag{17}$$

Substituting Eq.(17) in the last term of Eq. (16) and applying the divergence theorem to the first term on the right side of Eq.(17), we arrive at the final integral formulation of the advective side (AS).

$$\begin{aligned} AS = \int_\Gamma v_i(X)n_i(X)u^*(\xi;X)[u(x) - u(\xi)]d\Gamma - \int_\Omega v_i(X)u^*_{,i}(\xi;X)[u(X) - u(\xi)]d\Omega \\ - \int_\Omega v_{i,i}(X)u^*(\xi;X)[u(X) - u(\xi)]d\Omega. \end{aligned} \tag{18}$$

## 4 Numerical Tests

The physical domain of the problem and the boundary conditions are presented in Figure 1 (a), while the mesh discretization scheme and distribution of internal points are represented in Figure 1 (b). The domain has unitary dimensions, $L = 1$, the relationship $l_0/l = 0.5$ for the first row of internal points and the distance of double nodes $d = 0.02l_e$.

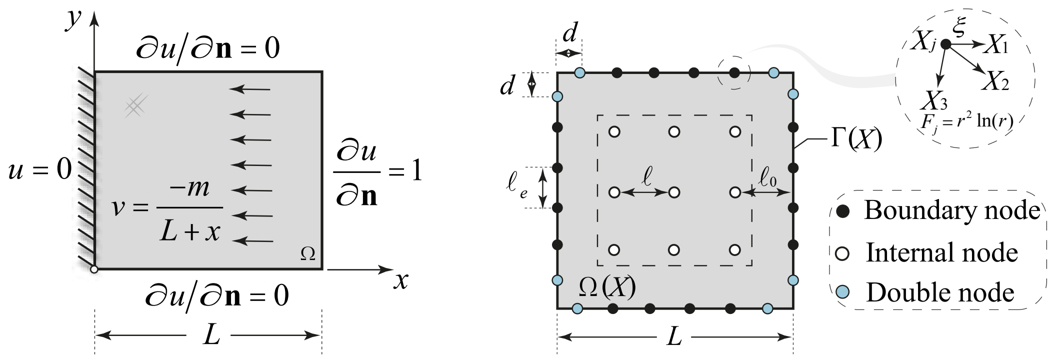


Figure 1. (a) Domain of Physical Problem (b) Mesh Distribution

Figure 2 shows the results obtained using a mesh with 80 boundary elements and 25 internal poles and compared the results of the relative errors when using the standard Gaussian integration and the auto-adaptive Telles's integration. Figure 2 (a) shows the results of the comparison between the quadratures using DIBEM formulation. The errors for the Telles's quadrature are slightly lower than those for the classical Gaussian quadrature for 4 and 8 Gauss points, from then on, the two integration schemes tend to have the same behavior. In Figure 2 (b) the DRBEM was used, and the self-adaptive integration only differs from the Gaussian quadrature for 4 Gauss points. The increase in the number of Gauss points is indifferent for larger amounts of Gauss points. Despite the self-adaptive quadrature quickly converging to the results of the classical quadrature, the DRBEM shows evidence of greater adherence with this type of special integration scheme [12].

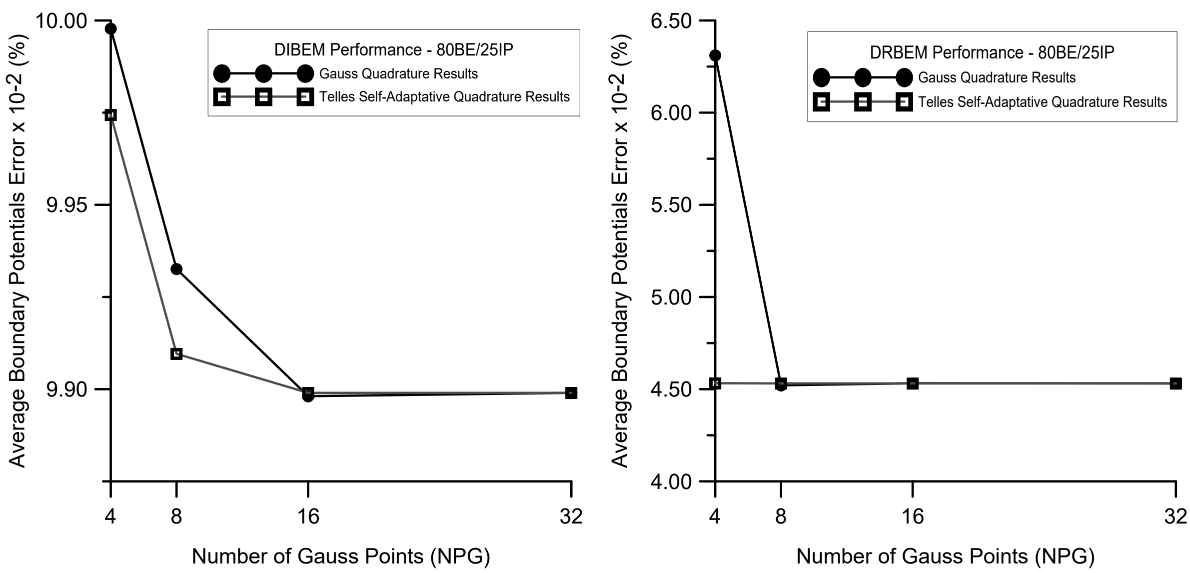


Figure 2. Parametric Analysis with NPG: (a) DIBEM (b) DRBEM

Figure 3 shows the simulations with variation of the advective parameter $m$. A mesh with 80 boundary elements was used and 81 internal poles were fixed. Despite numerical integration, 4 Gauss points were used and the results showed that the Telles's quadrature presented smaller errors in both simulations. It is worth noting that for DIBEM as shown in Figure 3 (a), the impact on the results were less expressive, when compared to the results with DRBEM presented in Figure 3 (b). For both integrations in DIBEM the errors were close, while in DRBEM the Telles's quadrature has greater impacts on the results. It is worth mentioning that the results for the DIBEM with the classical quadrature present lower errors than DRBEM's. This is probably due to a favorable amount and configuration of the internal poles[23].

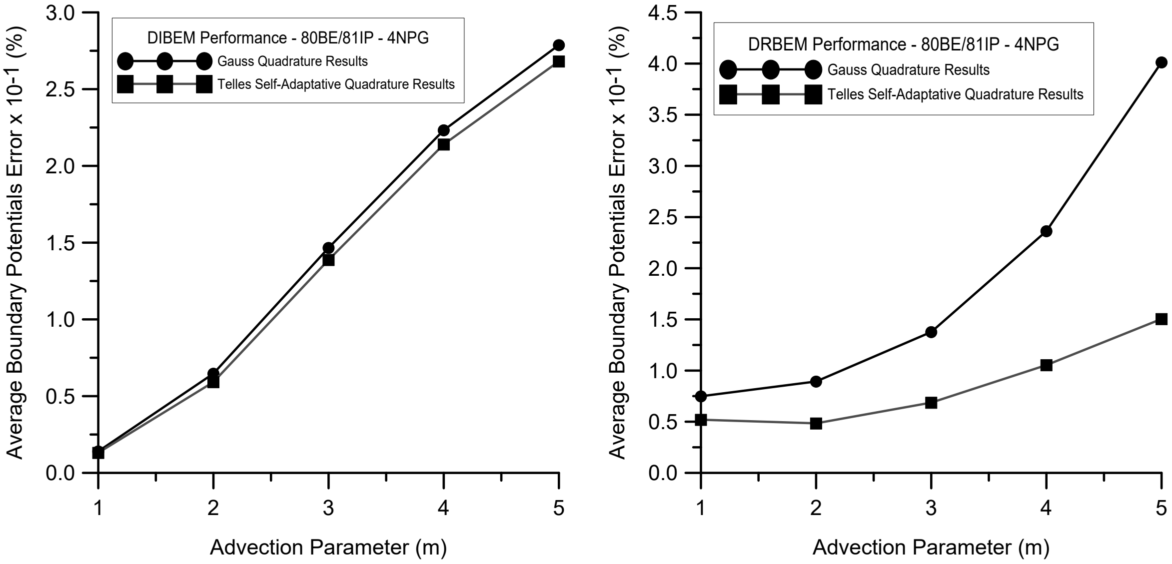


Figure 3. Parametric Analysis with Advection Parameter: (a) DIBEM (b) DRBEM

## 5 Conclusions

Telles' self-adaptive integration adopts a third-order coordinate transformation related to the Gaussian Quadrature points, where such transformation remains valid for any position of the singularity point and automatically produces a greater concentration of integration points close to the point of singularity. In this way, this special quadrature is able to adapt to the integrand, which tends to generate more accurate results.

In the context of BEM, a self-adaptive integration scheme is thought to solve singular and quasi-singular integrals. In this work it was observed that the self-adaptive integration is very effective using few Gaussian points, and when there is an increase in the number of points, it becomes similar to the standard Gaussian integration. The good adherence of Telles's quadrature with DRBEM, it's probably due to the presence of matrices [H] and [G] on the both sides of the final matricial system, improving the results when compared to those using standard Gaussian integration.

DIBEM presented a slight improvement in the results for the tested case, which raises the hypothesis of a non-effective synergy with Telles's quadrature. However, broader and more robust tests on the advective-diffusive model are necessary before inferring about some more general features. Furthermore, the application of this type of special quadrature, as well as others present in the literature, configures a potential line of research for the direct interpolation technique.

**Authorship statement.** The authors certify that they have participated sufficiently in the work to take public responsibility for the content, including participation in the concept, design, analysis, writing, or revision of the manuscript. Furthermore, each author certifies that this material or similar material has not been and will not be submitted to or published in any other publication before its appearance in CILAMCE 2022.

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