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A modified super-twisting algorithm with specified settling time: numerical investigation

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Abstract. The super-twisting algorithm (STA) is a finite-time stable algorithm that can be employed in control and observation of dynamical systems to ensure a good transient performance of tracking and estimation errors. However, only a conservative estimation of the convergence period can be obtained when using a classical STA. To address this limitation, the present work proposes two novel specified-time stable algorithms, in which the convergence instant can be directly specified as a system parameter. For the first one, we modify the right-hand side of an STA by replacing its non-smooth continuous term with a time-varying specified-time stabilizing function. For the second one, we alter the previously obtained system to recover the conventional STA performance after the specified period. We extensively analyze the algorithms' sensitivity to variations in each of their parameters through numerical simulations. With proper tuning, the last proposed dynamic system is shown to provide robust convergence to the origin at a specified instant of time.

Keywords: Finite-time stability, fixed-time stability, specified-time stability, second-order sliding modes, super-twisting algorithm.

1 Introduction

The super-twisting algorithm (STA), proposed by Levant [1] and further studied by a plethora of authors such as Davila, Polyakov, Moreno, and Nagesh [2–7], is a popular second-order sliding mode dynamic system that provides robust finite-time convergence to its state origin. However, distancing the initial states of said system indefinitely further from the equilibrium point causes the estimated upper bound of the convergence time to approximate infinity. A stronger form of finite-time stability, whose settling time estimation is a bounded function of the system's initial condition, was studied by Polyakov [8] and denoted in his work as fixed-time stability. Despite its advantage over finite-time stability, the settling time observed in simulations has shown to be a fraction of the settling time estimated bound, as illustrated in the works of Cruz-Zavala et al. [9] and Basin et al. [10]. To further improve on the aforementioned stability concepts, we claim that it is possible to devise an STA-like algorithm whose settling instant is a tunable parameter that explicitly appears in its mathematical model. To this new property, we have given the name specified-time stability.

Cruz-Zavala et al. [9] have developed a dynamic system based on the STA that provides fixed-time stability to its state origin. However, the estimate of the settling time is over a hundred times larger than the instant observed in the simulations, illustrating a conservatism of this estimation. An improvement of this method, using an adaptive STA-like algorithm, has been investigated by Basin et al. [11]. Their proposed method provides an estimated settling time bound closer to the value seen in their simulations, but their estimation considers that convergence is achieved once the states are driven to a vicinity of the state-space origin, rather than to the origin itself. Sánchez-Torres et al. [12, 13] and Jimenez-Rodriguez et al. [14, 15] have investigated the predefined-time stability property, which employs an exponential time-invariant stabilizing function containing the settling time upper bound as an explicit parameter. To the best of our knowledge, this property has not yet been investigated using a non-exponential stabilizing function, nor on an STA-like system.

To fill the aforementioned gaps in the literature, the present paper investigates the concept of specified-time stability using a polynomial time-varying stabilizing function on two modifications of an STA-like structure. The resulting systems are shown to present, with proper tuning of their parameters, finite-time convergence at a specified instant of time t_r, which explicitly appears as one of their parameters. The proposed algorithms are simulated in a variety of scenarios, to investigate their performance sensitivity to parameter variations. In particular, we also analyse the effects of altering the sampling time in numerical simulations and how this affects the chattering amplitude and the convergence of the system's states.

The remaining text is organized as follows. Section 2 formulates the specified-time modified STA algorithms and states the paper's problem. Section 3 evaluates the proposed algorithms under variation of their parameters using numerical simulations. Finally, Section 4 concludes the paper.

2 Problem Statement

Consider the super-twisting algorithm in the form provided by Moreno and Osorio [6]:

$$\dot{x}_1 = -k_1 \operatorname{sig}(x_1) + x_2, \tag{1}$$
$$\dot{x}_2 = -k_2 \operatorname{sign}(x_1) + \delta, \tag{2}$$

where $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$ are its state variables, $\operatorname{sig}(\xi) \triangleq |\xi|^{1/2} \operatorname{sign}(\xi)$, $k_1, k_2 > 0$ are tunable gains, and $\delta \in \mathbb{R}$ is an unknown, but bounded, disturbance. As the system in eqs. (1)–(2) has a discontinuous right-hand side, its solution will be understood in the sense of Filippov [16].

For this study, we propose two modifications to the STA. For the first one, named specified-time stable super-twisting algorithm (STSSTA), we alter the first equation of the algorithm, substituting its non-smooth continuous term as follows:

$$\dot{x}_1 = -\sigma(t, x) + x_2, \tag{3}$$
$$\dot{x}_2 = -k_2 \operatorname{sign}(x_1) + \delta, \tag{4}$$
$$\sigma(t, x) \triangleq \begin{cases} \frac{n}{t_r - t} x, & t \in [t_0, t_r), \\ g(x), & t \in [t_r, \infty), \end{cases} \tag{5}$$

where $n \in \mathbb{R}_{>1}$ is the specified time convergence rate, t_r is the specified settling instant, and $g(x) \triangleq 0$.

This novel algorithm, however, has a peculiarity. Once the instant t_r is reached, the first term in the right-hand side of eq. (3) becomes zero. From this instant forward, the behavior of $\dot{x}_1$ is guided by the behavior of the state x_2. Therefore, if by the time t_r, the state x_2 has not converged to zero, x_1 will leave zero, with no guarantees of reconvergence. To avoid dealing with the aforementioned peculiarity, we propose the second modification to the STA, named specified-time stable complementary super-twisting algorithm (STSCSTA). For this new algorithm, we assume $g(x) \triangleq k_1 \operatorname{sig}(x_1)$.

Under the STSCSTA dynamics, the states will behave as under the dynamics of the STSSTA while $t \in [t_0, t_r)$, and then assume the dynamics of the conventional STA throughout the remainder of the experiment. An example of the proposed algorithms' dynamics is illustrated by Fig. 1. Note that the blue line in Figs. 1b and 1d has not converged to zero at t_r. From this instant forward, the states oscillate around zero for the simulation using the STSSTA, whereas the STSCSTA is shown to drive both states to the origin.

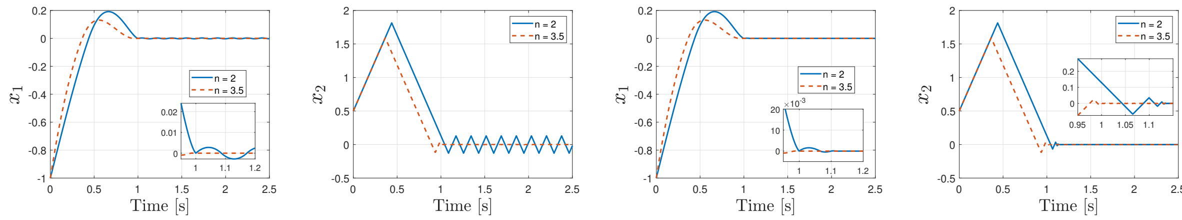
The purpose of this paper is to study the specified-time stability property of the system in eqs. (3)–(4), using both definitions of $g(x)$.

3 Simulation Analysis

Denote by $\Omega \triangleq (x_{1_0}, x_{2_0}, k_1, k_2, n, t_r, T_s)$ the vector containing the algorithms' parameters, where (x_{1_0}, x_{2_0}) are the states' initial values and T_s is the sampling period. In this section, we will thoroughly investigate how the variation of each of the parameters in Ω affects the proposed algorithms' performance.

3.1 On altering the sampling period

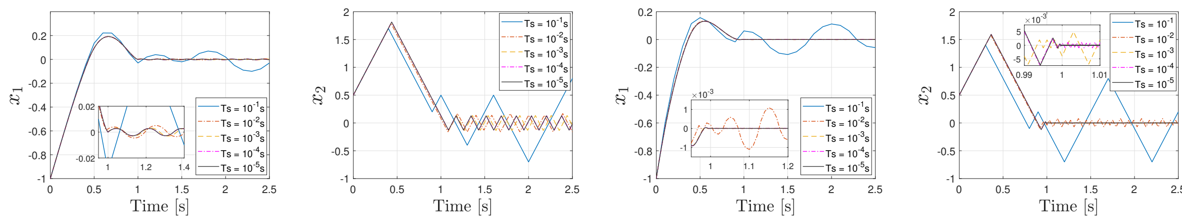
It is important to choose a suitable sampling period when analyzing algorithms with switching or time-varying terms, such as the STA and its proposed modifications. The existence of the switching term incurs an oscillating behavior on the system's states, denominated chattering. If T_s is poorly adjusted, the switching term will cause the excursion of the states to last longer. Therefore, although theoretically it can be proven that the states converge to zero for the conventional STA, in numerical simulations and practical applications the convergence of switching algorithms has to be understood as a convergence to an oscillating behavior around zero, whose amplitude is reduced when sampled at smaller intervals of time. Figures 2 and 3 demonstrate this aspect.



(a) (b) (c) (d)

Figure 1. Comparison of the STSSTA (Figs. a and b) and STSCSTA (Figs. c and d) algorithms, for two different convergence rates. Simulation parameters: $x_{i_0} = (-1,\ 0.5)$, $k_1 = 1$, $k_2 = 3$, $t_r = 1$s, $T_s = 10^{-4}$s and $\delta = 0$.

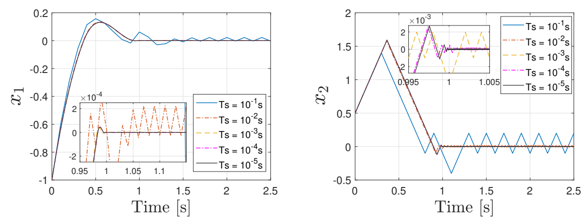
As seen in Fig. 1, for $n = 3.5$ the states converge to zero at the specified instant t_r. But Fig. 2d illustrates that the states converge to neighborhoods of the origin, whose sizes decrease as faster sampling rates are adopted. The same cannot be said about the behavior of the states in Fig. 2b, for which the small value of n does not provide the desired convergence, and increasing the sampling rate does not remove the steady-state oscillation.



(a) (b) (c) (d)

Figure 2. Simulation of the STSSTA algorithm for varying sampling periods and two different convergence rates. Simulation parameters: $x_{i_0} = (-1,\ 0.5)$, $k_2 = 3$, $t_r = 1$s, $n = 2$ for Figs. a and b, $n = 3.5$ for Figs. c and d, and $\delta = 0$.

A similar lasting steady-state oscillation can be seen in Fig. 3, when employing a slow sampling rate. Although the STSCSTA can force the states to zero even after t_r, the states are still affected by chattering and oscillate periodically. With larger values of T_s, as the period of time between signal switches lasts longer, the states drift inside a wider neighborhood of zero.



(a) (b)

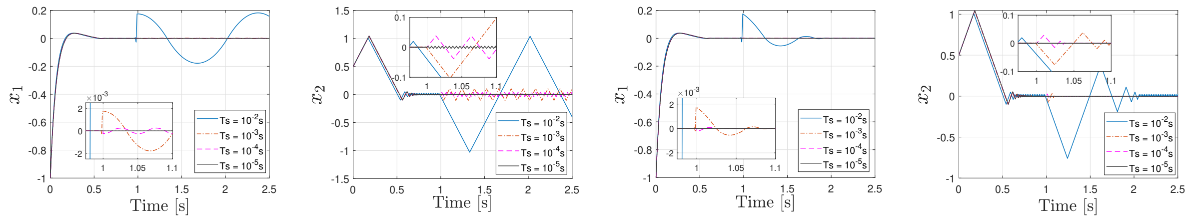
Figure 3. Simulation of the STSCSTA algorithm for varying sampling periods. Simulation parameters: $x_{i_0} = (-1,\ 0.5)$, $k_1 = 1$, $k_2 = 3$, $n = 3.5$, $t_r = 1$s and $\delta = 0$.

Another interesting aspect can be seen when simulating the system behavior with a high convergence rate parameter. For the simulations in Fig. 4, the output obtained for a sampling period of $T_s = 10^{-1}s$ was omitted, as it lead to divergences and jeopardized the graphs' readability.

The first term on the right-hand side of the equality in eq. (3) reaches very high values when t approximates t_r. The excitation of this discontinuity, when sampled at larger intervals, results in the leaps seen in the graphs.

From the previous simulations, we can conclude that these algorithms benefit from smaller sampling periods, as it also reduces chattering. In addition, the STSCSTA provides smaller chattering values than the STSSTA, demonstrating that the inclusion of the complementary term is beneficial in practical applications.

To standardize, the sampling time of $T_s = 10^{-4}$s will be adopted for the next simulations.

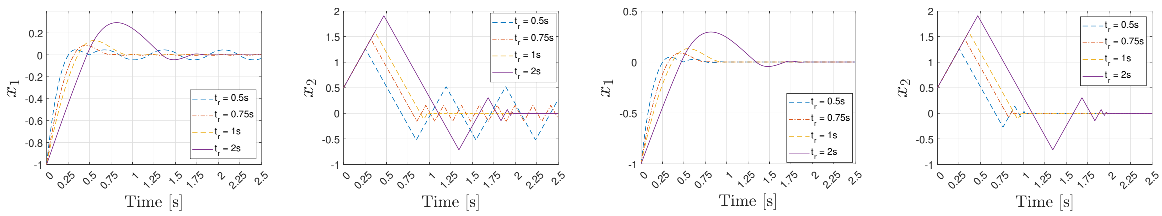


(a) (b) (c) (d)

Figure 4. Comparison of the STSSTA (Figs. a and b) and STSCSTA (Figs. c and d) algorithms for varying sampling periods. Simulation parameters: $x_{i_0} = (-1,\ 0.5)$, $k_1 = 1$, $k_2 = 3$, $n = 15$, $t_r = 1$s, and $\delta = 0$.

3.2 On altering the settling time

Figure 5 illustrates that fixed parameters n and k_2 do not provide finite-time convergence for every specified settling time t_r. It is also possible to see that, even in cases where the adopted parameters do not provide finite-time convergence for the STSSTA algorithm, the complementary switching term ensures this convergence. However, it can be said that the specified settling time restriction was not respected, since the states left zero for a short period after t_r.



(a) (b) (c) (d)

Figure 5. Comparison of the STSSTA (Figs. a and b) and STSCSTA (Figs. c and d) algorithms, for varying settling times. Simulation parameters: $x_{i_0} = (-1,\ 0.5)$, $k_1 = 1$, $k_2 = 3$, $n = 3.5$, $T_s = 10^{-4}$s and $\delta = 0$.

To further analyze this relationship between n and t_r, let us simulate the algorithms while modifying n for fixed values of reaching time. Due to the afore demonstrated additional benefits, this paper will hereinafter focus on the study of the STSCSTA.

3.3 On altering the convergence rate parameter

By varying the convergence rate parameter n, for different values of t_r, it is possible to see that smaller reaching times require faster convergence rates. To facilitate empirical results readability, let us assume that $n \in \mathbb{N}_{>1}$, and that convergence is reached once the oscillation of the states after t_r is only due chattering.

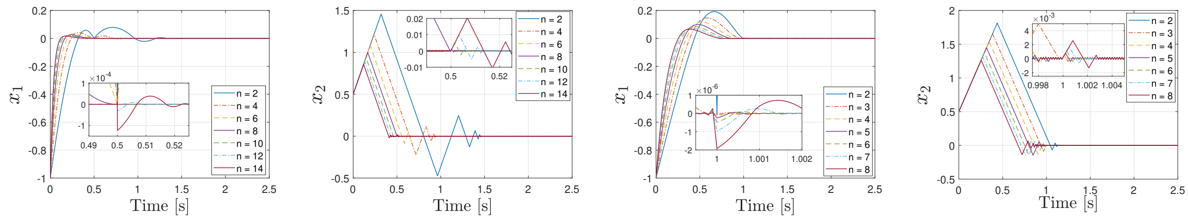
Figures 6a–6b show that, for a specified settling time of $t_r = 0.5$s, the states start to converge to zero for $n \geq 8$. Figures 6c–6d show convergence for $n \geq 3$ for a specified settling time of $t_r = 1$s. The complementary term forces the convergence in the cases that both states are not driven to zero at the specified instant. Figure 6 also illustrates that by increasing n, the excitation of the discontinuity for t close to t_r also increases, which results in a larger, although brief, deviation from the origin.

From the previous graphs, it is possible to conclude that, for fixed values of the other algorithm's parameters in Ω, smaller settling instants require larger values of n. Also, all values of n larger than the minimum necessary will too provide convergence. However, overdimensioning this parameter will increase the deviation of the states after the specified t_r. This effect can be mended by increasing the sampling rate, but it would require more computational power, which can be unfeasible in practical applications.

3.4 On altering the switching gains

The switching gain k_2 has two main roles in the proposed algorithms: to define the slope of x_2 oscillating behavior, and to provide state convergence despite of the influence of disturbances or uncertainties.

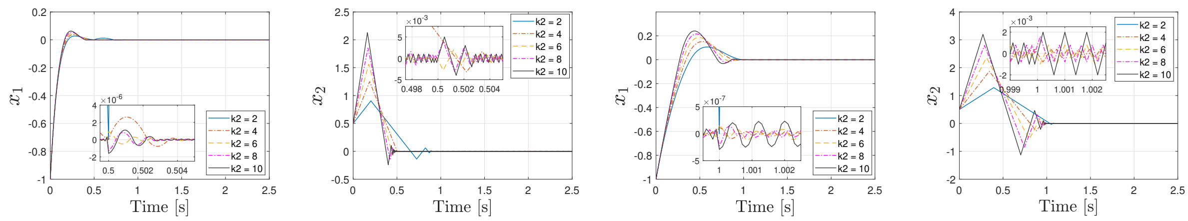
Figure 7 illustrates that k_2 has an important role on the convergence of the states. This can also be concluded



(a) (b) (c) (d)

Figure 6. Simulation of the STSCSTA algorithm for varying specified-time convergence rates. Simulation parameters: $x_{i_0} = (-1,\ 0.5)$, $k_1 = 1$, $k_2 = 3$, $T_s = 10^{-4}$s, $\delta = 0$, $t_r = 0.5$s for Figs. a and b, $t_r = 1$s for Figs. c and d.

when analysing eq. (4). As n controls how fast x_1 converges to zero, it also controls how fast the sign function switches. In addition, k_2 defines the variation rate of x_2 while the sign of x_1 remains unchanged. Therefore, properly choosing both these parameters is crucial to guarantee that x_2 converges to zero at $t \leq t_r$. By comparing Figs. 6b and 7b, it is possible to see how different values of k_2 affect the algorithm's performance for a fixed convergence rate $n = 6$. Once again, the STSCSTA is capable of forcing the convergence for every value of k_2 tested, although not always respecting the specified settling time restriction. For this algorithm, a proportional relationship between the switching gain and the chattering amplitude can be observed.



(a) (b) (c) (d)

Figure 7. Simulation of the STSCSTA algorithm for varying switching gains. Simulation parameters: $x_{i_0} = (-1,\ 0.5)$, $k_1 = 1$, $n = 6$, $T_s = 10^{-4}$s, $\delta = 0$, $t_r = 0.5$s for Figs. a and b, $t_r = 1$s for Figs. c and d.

The gain k_1 is used only in the STSCSTA algorithm. Its role is to mitigate the oscillation of the states, in cases where they have not converged to the origin in $t \leq t_r$ or when the excitation of the discontinuity in eq. (3) drives the states out of the origin. Also, larger values of k_1 yield faster convergences in said cases. For $k_1 = 0$, the STSCSTA behaves exactly as the STSSTA.

3.5 On altering the states initial values

From the previously defined vector Ω, the last parameters to be analyzed are the states' initial values x_{1_0} and x_{2_0}. Tables 1a and 1b summarize the data collected through simulation of different initial values. The tables show that, for a fixed value of k_2, the minimum value of n that provides convergence at t_r changes according to the initial states. Moreover, when increasing the value of k_2, these minimum values of n reduce significantly.

Table 1. Sensibility of the algorithm to variation of initial conditions, measured by obtaining the smallest value of n that provides convergence for different values of switching gain.

$\mathbf{x_{2_0}}$ \ $\mathbf{x_{1_0}}$	**-10**	**-5**	**-1**	**0**	**1**	**5**	**10**
-2.5	4	4	4	4	30	54	64
-0.5	6	5	3	3	3	7	9
0	7	6	3	3	3	6	7
0.5	9	7	3	3	3	5	6
2.5	64	54	30	4	4	4	4

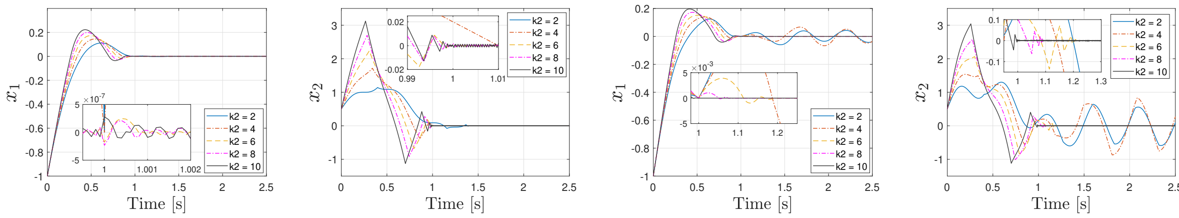
(a) $k_2 = 3$

$\mathbf{x_{2_0}}$ \ $\mathbf{x_{1_0}}$	**-10**	**-5**	**-1**	**0**	**1**	**5**	**10**
-2.5	4	4	4	3	4	6	8
-0.5	5	4	4	3	4	5	6
0	5	4	4	3	4	4	5
0.5	6	5	4	3	4	4	5
2.5	8	6	4	3	4	4	4

(b) $k_2 = 8$

3.6 On introducing a bounded disturbance

Let us define a bounded differentiable disturbance signal $\delta = \rho \cos(4\pi t)$. As previously mentioned, the switching gain k_2 is also responsible to counteract the effect of external disturbances or model uncertainties. To analyze the obtained response for a disturbed system, using different ρ values, let us assume that robustness is reached once there is no visual influence of the applied disturbance on the response of x_i after the specified t_r.



(a) (b) (c) (d)

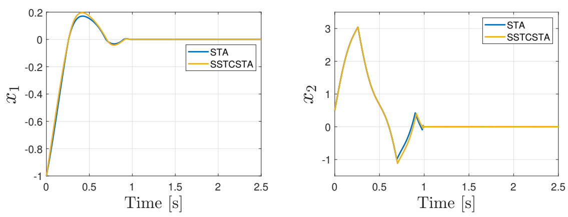
Figure 8. Simulation of the STSCSTA algorithm for varying switching gain and two different disturbance bounds. Simulation parameters: $x_{i_0} = (-1,\ 0.5)$, $k_1 = 1$, $n = 3.5$, $t_r = 1$s, $T_s = 10^{-4}$s, $\rho = 2.$ for Figs. a and b, and $\rho = 5$ for Figs. c and d.

Figure 8 illustrates that larger values of switching gain k_2 are required in order to confer to the system robustness to larger disturbances. Also, even in cases that the disturbance upper bound is smaller than k_2, there is the possibility that it forces x_2 to assume a value different than zero at t_r, causing the states to be forced to zero after the specified settling instant, as seen in previous examples. It is important to remark that, when affected by bounded disturbances with an offset greater than k_2, the states will diverge, as the right-hand side of eq. (4) will not change its signal throughout the experiment.

3.7 On comparing to the conventional STA

The original STA does not provide to the user an explicit reaching instant, nor can it be easily calculated from its switching gains. Theorems that prove the algorithm's finite-time convergence only specify an upper bound for the reaching time, but cannot pinpoint the exact instant of convergence.

For the simulations illustrated in Fig. 9, the values of k_1 and k_2 were carefully chosen (through trial and error), to provide convergence at exactly t_r for both methods. The obtained results are very similar, but determining the switching gain that provided convergence at t_r for the STA was an extenuating task, whereas it is defined only by a parameter for the STSCSTA.



(a) (b)

Figure 9. Simulation comparing STSCSTA and STA. Simulation parameters: $x_{i_0} = (-1,\ 0.5)$, $k_1 = 2.737$, $k_2 = 10$, $n = 3.5$, $t_r = 1$s, $T_s = 10^{-4}$s, and $\rho = 5$.

4 Conclusions

The modified super-twisting algorithms provide the desired finite-time convergence at a specified instant of time with the proper choice of the parameters in the vector Ω, as shown by the simulations. The introduction of the complementary term grants robust stability to the system after t_r, while also reducing the chattering effect. The state x_1 converges empirically to zero at t_r for any $n > 1$, whereas the behavior of x_2 depends on the values of n and k_2. Large values of n cause the states to briefly diverge at t close to t_r, whereas large values of k_2 incur in larger chattering. Therefore, it is beneficial to choose the smallest values of these parameters that still yield the desirable convergence. For future works, we intend to define an equation that calculates n and k_2 that provide the desired performance, based on the other values in Ω and the disturbance upper bound, as well as to investigate the application of these algorithms for states observation.

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