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An analytical model for stresses induced by pore-water pressures in porous rock slabs

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Abstract. A well-known issue of marble slabs casted in building façades is the phenomenon of bowing, which may cause damage to constructions all over the world. Laboratory studies have shown that some kinds of marble subjected to temperature oscillations suffer from non-uniform and permanent expansion, and that this problem is exacerbated in saturated environment. The understanding of this pathology is still incomplete, despite its importance for civil construction, in order to prevent or even avoid this issue. In the formulation developed herein, heat transfer problem is modeled by sol-air formulation, in order to linearize irradiation transfer and convective transfer. The fully linearized problem is then solved by trigonometric series specially developed for time-periodic problems in order to represent the daily cycles of heating and cooling. The analytical solution developed is compared to a formulation previously developed by one of the authors of this work (Ito), extended to saturated environment under undrained conditions. The result of this study is a contribution to the understanding of bowing and demonstrates that the development of pore pressures in undrained conditions can accelerate the degradation of marble.

Keywords: thermally induced stresses ; elasticity ; pore pressures ; rock mechanics

1 Introduction

One of the key mechanisms responsible to cause degradation in building materials used in external environments is natural weathering. The physical degradation due to daily variation of temperature in metamorphic rocks has been considered an important degradation mechanism, and it has incorporated the international standards only a few decades ago. A phenomenon known as *bowing* can cause intragranular decohesion of calcite and induce large deformations in marble slabs and significant loss of strength after long exposure periods. In this context, thermoelastic analysis of plates used in marble facade coverings are presented [1].

Ito's work outputs [1] indicate a greater influence of the changing in the thermal gradient through slab thickness rather than the absolute value of the thermal gradient between the surfaces on the induced stresses due to thermal variation. In other words, the heat transient generates a temperature profile that is not linear and only this non-linear temperature profile can induce thermal stresses in slabs [1].

A research made by Koch and Siegesmund [2] verified the influence of humidity in bowing tests and compared with research in dry conditions initially made by S. Battaglia and Mango [3] . According to this study, in dry condition and after some heating cycles, the residual strain will not increase anymore. The degradation is greater at the beginning but then it stagnates. However, in a wet environment the deterioration may increase progressively, so the bowing phenomenon continues to grow continuously [2].

Ito started a theoretical quantitative study of the stresses for the case of dry marble (drained conditions) [1]. In this paper, the stress evolution for undrained environment will be estimated. Biot's theory will be used, that is, it will be considered that there can be volume variation of solids and fluids in undrained condition, which causes matrix volume changes. However, the formulation of linear poroelasticity will be based on the formulation of micromechanics exposed by Cheng [4], because it uses more usual parameters for the areas of soil and rock mechanics than the original formulation written by Biot and Willis [5], even though they are equivalent in terms of results.

2 Heat transfer and thermal stresses formulation

This work follows closely the formulation developed by Ito et al. [1, 6], which considers the heat transfer within the rock slab as governed by the heat conduction equation in one dimension (the slab thickness), that is

$$\frac{\partial T}{\partial t} = \alpha_t \frac{\partial^2 T}{\partial x^2}, \tag{1}$$

where T is the temperature in any point of the rock, t is the time elapsed since the initial condition, x is the coordinate transverse to the slab and α_t is the thermal diffusivity, which is given by

$$\alpha_t = \frac{k_t}{\rho_c c_p}, \tag{2}$$

where k_t is the thermal conductivity of the porous medium (solids and fluids), ρ_c is the density and c_p is the specific heat capacity of the same matrix.

The whole heat exchange of the slab with the external environment is approximated by a linear formulation known as Newton's law of cooling, that is

$$q''_{c,i} = (T_i - T_{w,i})\, h_i, \qquad q''_{c,o} = (T_o - T_{w,o})\, h_0, \tag{3}$$

where $q''_{c,i}$ is the heat flux in the internal slab surface (W/m^2 in SI units), T_i is the internal environment temperature, $T_{w,i}$ is the temperature of the internal slab surface, $q''_{c,o}$ is the heat flux in the external (outside) surface of the slab, T_o is the external environment temperature, $T_{w,o}$ is the temperature of the external surface of the slab. Although the original formulation of Newton's law accounts for a linearization of convective heat transfer, in the context of buildings, T_o and T_i may be recalculated in order to take into account solar heat gains (sol-air temperature), along with other heat exchanges [7].

Ito et al. [6] consider that T_i and T_o are almost periodic ("circadian"), which allows T_i and T_o to be approximated by a trigonometric (Fourier) series in time and a series of products of trigonometric and exponential functions in space. This solution seemed to be proposed initially by Alford et al. [8] and it is presented here an equivalent formulation which was proposed by Ito [1]. Hence, this periodic solution may be calculated as

$$T(x,t) = \sum_{n=1}^{\infty}\sum_{i=1}^{4} X_n^{(i)} T_n^{(i)}(x,t), \tag{4}$$

where

$$\begin{aligned}
T_n^{(1)}(x,t) &= ShC_n(x)\cos(\omega_n\, t) - ChS_n(x)\sin(\omega_n\, t) && (5)\\
T_n^{(2)}(x,t) &= ShC_n(x)\sin(\omega_n\, t) + ChS_n(x)\cos(\omega_n\, t) && (6)\\
T_n^{(3)}(x,t) &= ChC_n(x)\cos(\omega_n\, t) - ShS_n(x)\sin(\omega_n\, t) && (7)\\
T_n^{(4)}(x,t) &= ChC_n(x)\sin(\omega_n\, t) + ShS_n(x)\cos(\omega_n\, t) && (8)
\end{aligned}$$

are solutions of the heat conduction equation (1), defined with aid of the following auxiliary functions:

$$\begin{aligned}
ShC_n(x) &= \sinh\left(\sqrt{\frac{\omega_n}{2\alpha_t}}x\right)\cos\left(\sqrt{\frac{\omega_n}{2\alpha_t}}x\right), \qquad ChS_n(x) = \cosh\left(\sqrt{\frac{\omega_n}{2\alpha_t}}x\right)\sin\left(\sqrt{\frac{\omega_n}{2\alpha_t}}x\right),\\
ChC_n(x) &= \cosh\left(\sqrt{\frac{\omega_n}{2\alpha_t}}x\right)\cos\left(\sqrt{\frac{\omega_n}{2\alpha_t}}x\right), \qquad ShS_n(x) = \sinh\left(\sqrt{\frac{\omega_n}{2\alpha_t}}x\right)\sin\left(\sqrt{\frac{\omega_n}{2\alpha_t}}x\right).
\end{aligned} \tag{9}$$

Table 1. Thermomechanical properties of Carrara marble [10]

Property	Value	Property	Value
Young modulus E	52.4 GPa	Diffusion coefficient (α_t)	0.0118 cm^2/s
Poisson's ratio ν	0.16	Bulk thermal expansion coefficient (β)	$17.7{\cdot}10^{-6}$ C^{-1}

Table 2. Thermomechanical properties of calcite [11]

Property	Value
Thermal Conductivity	5.526 (para) 4.646 (perp) W m^{-1} K^{-1} at 273K
Thermal Expansion	25 (para) -5.8 (perp) $\cdot$ 10^{-6}/°C at 273K
Specific Heat Capacity	852 J Kg^{-1} K^{-1}
Young's Modulus	72.35 (perp) 88.19 (para) GPa
Shear Modulus	35 GPa
Bulk Modulus	129.53 GPa
Elastic Coefficients	C11=137; C12=45; C13=45; C14=(21); C33=79

$X_n^{(i)}$ are the coefficients of these functions in the series, and they are calculated as functions of the boundary conditions at the internal and external surfaces of the slab [1]. One should bear in mind that this series describes a periodic heat flow, to which there is no initial conditions, but only the periodic boundary conditions at the surfaces $x = 0$ and $x = L$. It can be any combination of Dirichlet, Neumann or mixed condition, with the only exception of two Neumann periodic conditions, for which the problem remains undetermined.

After defining the temperature field $T(x, t)$, the thermally induced longitudinal stresses in the slab can be calculated using the formulation proposed by Johns [9]. This solution supposes a linear elastic medium, which, for plane strain condition, reads

$$\sigma_{yy} = \frac{\beta}{3}\frac{E}{1-\nu}\left[-T(x,t) + \frac{1}{L}\int_0^L T(x,t)\,\mathrm{d}x + \frac{12}{L^3}\left(x - \frac{L}{2}\right)\int_0^L T(x,t)\left(x - \frac{L}{2}\right)\mathrm{d}x\right], \tag{10}$$

where β is the volumetric thermal expansion coefficient, E is the Young modulus of the material, ν is its Poisson modulus and L is the thickness of the slab.

3 Poroelastic parameters

Table 1 presents some thermomechanical properties of Carrara Marble (bulk porous material) collected by Ferrero et al. [10], while Tabel 2 presents some properties of calcite crystal (grain fabric), as collected by Crystran Ltd.

The *drained bulk modulus* K of the porous matrix is defined as

$$K = -\left.\frac{\Delta P}{\Delta V/V}\right|_{\text{drained}}, \tag{11}$$

where P is the compressive stress, V is the volume of the porous material specimen and ΔP and ΔV are their respective increments. This parameter is, together with the shear modulus G, directly determined by the usual (drained) parameters E (Young's modulus) and ν (Poisson's ratio) by equations

$$K = \frac{E}{3\,(1-2\nu)}, \quad G = \frac{E}{2\,(1+\nu)}. \tag{12}$$

Table 3. Coefficients for calculus of water density [12]

Coefficient	Value	Coefficient	Value
$a_1/°\text{C}$	-3.983035± 0.00067	a_5/(kg m^{-3})	999.974950 ± 0.00084
$a_2/°\text{C}$	301.797	k_0/(10^{-11} Pa^{-1})	50.74
$a_3/°\text{C}^2$	522 528.9	k_1/(10^{-11} Pa^{-1})	-0.326
$a_4/°\text{C}$	69.348 81	k_2/(10^{-11} Pa^{-1})	0.00416

In the undrained condition, as fluid cannot move in the porous medium, its stiffness must be taken into account. Moreover, its fluid-solid interaction will change solid behavior. The *fluid bulk modulus* is defined as [4]

$$K_f = -\left.\frac{\Delta p}{\Delta V_f/V_f}\right|_{\text{drained}}, \tag{13}$$

where p is the pore fluid nanometric pressure and V_f is the volume of fluids in the porous medium. In the present case, the saturated medium in undrained case, the fluid occupies the entire pore space. According to Tanaka et al. [12], water density can be calculated as

$$\rho = a_5\left[1-\frac{(T+a_1)^2(T+a_2)}{a_3(T+a_4)}\right]\left[1+(k_0+k_1T+k_2T^2)p\right], \tag{14}$$

where the coefficients a_i and k_i are presented in Table 3. This formulation allows calculation of the fluid bulk modulus as

$$K_f = \frac{\rho}{\partial\rho/\partial p} = \frac{1+(k_0+k_1T+k_2T^2)p}{k_0+k_1T+k_2T^2} \tag{15}$$

It also allows calculation of the coefficient of volumetric thermal expansion of water as

$$\beta_f = -\frac{\partial\rho/\partial T}{\rho} = -\frac{(T+a_1)\left[(T+a_1)(T+a_2)-(3T+2a_2+a_1)(T+a_4)\right]}{(T+a_4)\left[a_3(T+a_4)-(T+a_1)^2(T+a_2)\right]} - \frac{(k_1+2k_2T)\,p}{1+(k_0+k_1T+k_2T^2)\,p}. \tag{16}$$

According to Cheng [4], in a compressibility test with constant effective stress p' ($\Delta p' = 0$), the *bulk modulus of the solid phase* may be calculated as

$$K_s = -\frac{\Delta p}{\Delta V_s/V_s}, \tag{17}$$

where V_s is the solid phase volume and ΔV_s is its increment (it should be emphasized that this is not a formal definition of K_s). Following Cheng [4], the linear theory of poroelasticity would predict the existence of two extra bulk moduli for solid phase.The present study assumes that all three moduli K_s, K_s' and K_s'' are equal, which simplifies the formulation as a whole. This ideal model is, in fact, more adequate to a matrix with homogeneous distribution of isotropic grains, according to the aforementioned author. Nevertheless, it is supposed that the magnitude of the results won't change significantly if future experimental data support any difference between the three moduli. The undrained bulk modulus for an *ideal porous medium* is calculated as [4]

$$K_u = -\left.\frac{\Delta P}{\Delta V/V}\right|_{\text{undrained}} = K + \frac{K_f\,(K_s-K)^2}{K_f\,(K_s-K)+n\,K_s\,(K_s-K_f)}, \tag{18}$$

where n is the porosity, which, for a saturated porous medium, is

$$n = \frac{V_f}{V}. \tag{19}$$

Still, according to the same author, the undrained coefficient of volumetric thermal expansion is

$$\beta_u = (1 - n\,B)\,\beta_s + n\,B\beta_f, \tag{20}$$

where β_s is the coefficient of volumetric thermal expansion of the solids, while β_f is the coefficient of volumetric thermal expansion of the fluid and

$$B = \frac{\Delta p}{\Delta P} = 1 - \frac{n\,K\,(K_s - K_f)}{K_f\,(K_s - K) + n\,K\,(K_s - K_f)} \tag{21}$$

is the *Skempton pore pressure coefficient*.

Having K_u and G in hands, one can calculate the equivalent Young modulus and Poisson ratio for an undrained porous material:

$$E_u = \frac{9K_u\,G}{3K_u + G}, \quad \nu_u = \frac{3K_u - 2G}{2\,(3K_u + G)}. \tag{22}$$

In order to calculate the thermal total stress in a slab in undrained conditions, one should substitute E, ν and β by E_u, ν_u and β_u in (10).

The pore pressure may be calculated by

$$p = M\,(-\alpha e + \zeta + \beta_e T)\,, \tag{23}$$

where

$$M = \frac{K_f\,K_s^2}{K_f\,(K_s - K) + n\,K_s\,(K_s - K_f)}, \quad \alpha = 1 - \frac{K}{K_s'}, \quad \beta_e = \alpha\beta_d + \beta_u, \tag{24}$$

α is the Biot coefficient, e is the volumetric strain and ζ is the variation in fluid content, defined as the amount of fluid volume entering the solid frame per unit volume of solid frame; for ideal porous medium, $\beta_d = \beta_s$ [4].

4 Results and discussion

The formulation presented prevously was applied to a real case of the southern facade of the Pescara Justice Court, which was monitored by Ferrero et al. [10] in 2007, in order to evaluate bowing problems. The external facade was made of 3 cm thick Carrara marble slabs, and this study will focus on the condition estimated at 14:00 on August 10$^{\text{th}}$, that was estimated by Ito et al. [6] as the instant in which a sudden fall of temperature caused a peak of thermal stresses. The values of $E_u = 61 \cdot 10^9$ Pa and $\nu_u = 0.35$ were found from equation 22 with data from tables 1, 2 and 3. Following Hebhoub et al. [13], the porosity of the marble was assumed as the approximate value of 2%. The stresses calculated for the *undrained* condition by the formulation presented in this work was compared with the stresses calculated in *drained* condition by Ito et al. [6]. Therefore, the analytical solution developed in this article was compared with the implementation of the same problem in finite elements developed by ITO and applied to undrained conditions [1] in Figure 2. It is observed that, while the analytical

[1] available in `https://github.com/guimaraesyc/Undrained-Solution.git`

solution considers an Euler-Bernoulli beam (virtually, an infinite beam), the finite element implementation deals with a two-dimensional plate (in plane strain state), with a length of 1.05 meters and same thickness. As the results are very close, both formulations are valid as estimators of stresses and pore pressures induced by non-uniform temperature gradients.

Figure 1 presents longitudinal (σ'_y) and transversal (σ'_x) effective stresses as calculated in drained and undrained conditions, with drained condition being associated to the dry marble and undrained condition being associated to the saturated marble.

Some observations about these results are worthwhile:

- Total stress $\sigma'_y + p$ in the undrained (”wet”) case is more than twice as high as in the drained (”dry”) case.
- Effective longitudinal stress (σ'_y) is approximately 50 percent higher in undrained case than in drained case.
- The (undrained) pore pressures (p) are of the same magnitude as the longitudinal stresses in the drained case.
- In the undrained case, total longitudinal stress is more than twice as high as in the drained case and the effective longitudinal stress is approximately 50 percent higher than in the drained case.
- Pore pressures of the undrained case have the same magnitude as the longitudinal stresses in the drained case. These pore pressures are higher near to the surface of the plate.

It should be noted that the undrained pore pressure profile includes non-zero pressures at the surface of the plate. Considering the effective porosity of the marble, i.e., the interconnected pores, the fluid pressure close to the surface can only be high in the specific case where the deformations induced by thermal loads occur much faster than the dissipation of the pore pressures generated. Judging by the low hydraulic conductivity of marble, it is believed that this may be the case and the pore pressure profile found in this work should be a good approximation of the real profile for moderate to large distances from the external surface.

In the present case (undrained), the pore pressures cause only tensile stresses near the surface, in the transverse direction. Therefore, this model might explain the degradation by disaggregation of microscopic calcite crystals on the surface of the plate.

5 Conclusion

This study showed that the development of pore pressures in undrained conditions can accelerate the degradation of Carrara Marble by heat transitions, when compared to the drained situation, as developed by [1]. The theoretical model presented here shows that water can release long-term stresses that do not exist in drained environment. Besides that, the transverse stresses are, in fact, greater in the undrained case.

This study was performed according to the ideal porous model [4], that is, isotropic and homogeneous grains.

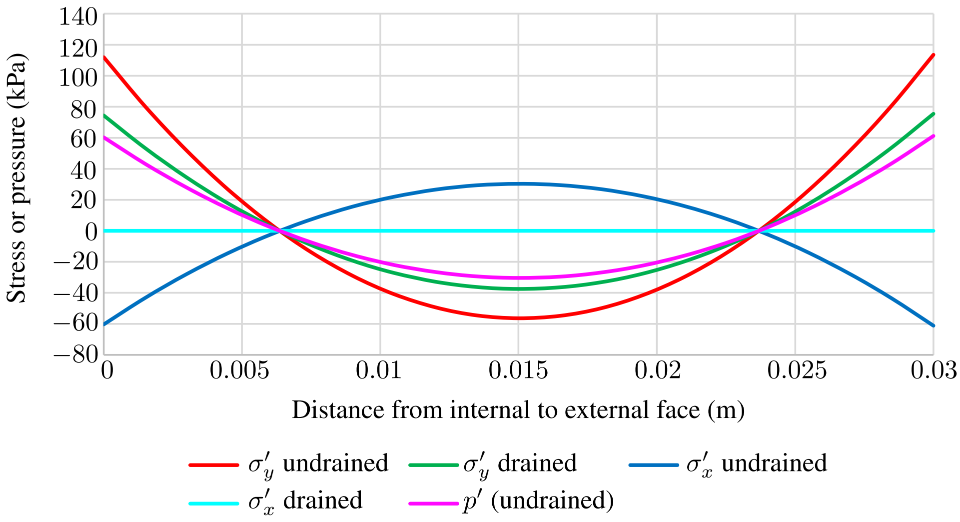

Figure 1. Thermal stresses calculated at 14:00 on August 10th for dranied [6] and undrained (present study) cases

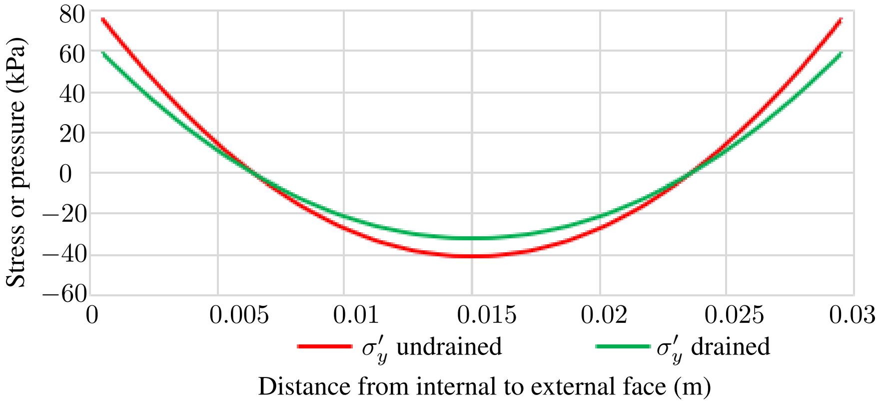

Figure 2. Finite element solution developed by [1] applied for drained and undrained conditions

Therefore, the study uses an equivalent content model that does not take into account specific microscopic effects, such as anisotropy, heterogeneity and stress localisation. For the time being, as shown, the degradation of the structure generates tensile stresses, which already explains bowing; therefore it should be considered as a contribution the understanding and prevention of the bowing. Future studies should address these effects for a more specific understanding of the bowing phenomenon.

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