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Analytical model for dissipation of thermally induced pore pressure in marble slabs

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Abstract.
This paper presents a thermomechanical model of a saturated porous slab, which is suitable to the study of rocks subjected to thermal exchanges with the surrounding environment of a building. This model involves linear differential equations of stress equilibrium and transient pore pressure dissipation and its main purpose is to help to the understanting of the *bowing*, which is an important phenomenon that occur when some porous rocks are subject to temperature cycles. Moreover, an analytical solution for a steady state condition for pore pressure is developed, which describes the situation where the generation of pressure due to cooling of a slab is compensated by pressure dissipation caused by water loss (consolidation). This analytical solution should be useful to test future numerical implementations of the thermoelasticity model presented herein.

Keywords: Biot Consolidation, Carrara Marble, Bowing

1 Introduction

The degradation of building envelope materials subjected to wheathering is a well known matter of concern in civil construction all over the world. In particular, a phenomenon known as *bowing* occurs in some marble slabs subjected to daily temperature oscilations and this degradation seems to be enhanced when the marble is saturated [1]. Ito et al. [2] started an analytical study on thermally induced stresses in dry marble slabs subjected to periodic temperature oscilations. Guimarães et al. [3] developed an "upper bound" estimation of pore pressure development by adapting the solution of Ito et al. to undrained conditions. In this work, it is presented a simple solution for dissipation of pore pressure in porous slabes subjected to a sudden temperature fall. The material is considered as a porous linear elastic medium, in which thermal conduction is ruled by Fourier law and water flow in ruled by Darcy's law.

In the next section, it is presented the framework of thermoporoelasticity as it will be used in this paper. The third section is dedicated to present the simple heat transfer solution to be used. Following that, the dissipation solution is developed and commented.

2 Anisotropic formulation of thermoelasticity

For transverse isotropy, the z-axis is assigned as the axis of material rotational symmetry. For plane strain state, the constitutive relations can be expressed as [4, 5]

$$\sigma_{xx} = M_{11}\, e_{xx} + M_{13}\, e_{zz} - \alpha_1\, p - \alpha_{d,1}\, T, \quad (1)$$
$$\sigma_{yy} = M_{12}\, e_{xx} + M_{13}\, e_{zz} - \alpha_1\, p - \alpha_{d,1}\, T, \quad (2)$$
$$\sigma_{zz} = M_{13}\, e_{xx} + M_{33}\, e_{zz} - \alpha_3\, p - \alpha_{d,3}\, T, \quad (3)$$
$$\sigma_{xz} = 2M_{55}\, e_{xz}, \quad (4)$$

where σ_{ij} are the components of the stress tensor (tensile normal components are positive), e_{ij} are the components of the strain tensor, M_{ij} are the elastic constants of the drained stress-strain relationship, p is the pore water pressure (compression is positive) and T is the temperature, which should be "gauged" for $T = 0$ when null strains produce null stress. The following drained thermophysical properties may be calculated:

$$\alpha_{d,1} = M_{11}\, \alpha_{t,1} + M_{12}\, \alpha_{t,1} + M_{13}\, \alpha_{t,3}, \qquad \alpha_{d,3} = 2M_{13}\, \alpha_{t,1} + M_{33}\, \alpha_{t,3}, \quad (5)$$

where $\alpha_{t,1}$ and $\alpha_{t,3}$ are the linear thermal expansion coefficients; α_1 and α_3 are the anisotropic effective stress coefficients, which may be calculated as

$$\alpha_1 = 1 - \frac{M_{11} + M_{12} + M_{13}}{3K_s}, \qquad \alpha_2 = 1 - \frac{2M_{13} + M_{33}}{3K_s}, \quad (6)$$

where K_s is the bulk modulus of solid phase. The last constitutive equation is useful in calculation of fluid pore pressure (still in plane strain state),

$$p = M(-\alpha_1\, e_{xx} - \alpha_3\, e_{zz} + \zeta + \beta_e\, T), \quad (7)$$

where ζ is the variation of fluid content caused by flow. For an *ideal porous medium* [5], one gets

$$\beta_e = \alpha\beta_s + \beta_u, \quad (8)$$

where $\beta_s = 2\alpha_{t,1} + \alpha_{t,3}$ is the volumetric thermal expansion coefficient of the solid phase and β_u is the undrained volumetric thermal expansion coefficient, which is calculated as

$$\beta_u = (1 - nB)\, \beta_s + nB\beta_f. \quad (9)$$

Here, n is the porosity, β_f is the volumetric thermal expansion coefficient of the fluids, α is the isotropic Biot effective stress coefficient given by

$$\alpha = 1 - \frac{\bar{M}}{K_s}, \quad (10)$$

$\bar{M}$ is the "average" bulk modulus of the matrix given by

$$\bar{M} = \frac{2M_{11} + M_{33} + 2M_{12} + 4M_{13}}{9}, \quad (11)$$

and B is the *Skempton pore pressure coefficient* given by [4, 5]

$$B = 1 - \frac{n\,\bar{M}\,(K_s - K_f)}{K_f\left(K_s - \bar{M}\right) + n\,\bar{M}\,(K_s - K_f)}, \tag{12}$$

M (Biot modulus) is the inverse of the storage coefficient, given by [4, 5]

$$M = \frac{K_f\,K_s^2}{K_f\left(K_s - \bar{M}\right) + n\,K_s\,(K_s - K_f)}. \tag{13}$$

Equations (1) and (3) can be used to calculate e_{xx} and e_{zz}, which read

$$e_{xx} = -\frac{M_{13}}{M_{11}M_{33} - M_{13}^2}\sigma_{zz} + \frac{M_{33}\alpha_1 - M_{13}\alpha_3}{M_{11}M_{33} - M_{13}^2}p + \frac{M_{33}\alpha_{d,1} - M_{13}\alpha_{d,3}}{M_{11}M_{33} - M_{13}^2}T, \tag{14}$$

$$e_{zz} = \frac{M_{11}}{M_{11}M_{33} - M_{13}^2}\sigma_{zz} + \frac{M_{11}\alpha_3 - M_{13}\alpha_1}{M_{11}M_{33} - M_{13}^2}p + \frac{M_{11}\alpha_{d,3} - M_1 3\alpha_{d,1}}{M_{11}M_{33} - M_{13}^2}T, \tag{15}$$

With the previous equations, eq. (7) can be used to calculate ζ, which reads

$$\zeta = \frac{\alpha_3 M_{11} - \alpha_1 M_{13}}{M_{11}M_{33} - M_{13}^2}, (\sigma_{zz} + A_1 p + A_t T), \tag{16}$$

$$A_1 = \frac{M_{33}\alpha_1^2 - 2M_{13}\alpha_1\alpha_3 + M_{11}\alpha_3^2}{\alpha_3 M_{11} - \alpha_1 M_{13}} + \frac{M_{11}M_{33} - M_{13}^2}{M\,(\alpha_3 M_{11} - \alpha_1 M_{13})}, \tag{17}$$

$$A_t = \frac{M_{33}\alpha_1\alpha_{d,1} - M_{13}\alpha_1\alpha_{d,3} + M_{11}\alpha_3\alpha_{d,3} - M_{13}\alpha_3\alpha_{d,1}}{\alpha_3 M_{11} - \alpha_1 M_{13}} - \frac{M_{11}M_{33} - M_{13}^2}{\alpha_3 M_{11} - \alpha_1 M_{13}}\beta_e. \tag{18}$$

According to Abousleiman et al. [4], there is a relation between the drained and the undrained moduli, which reads

$$M_{ij}^u = M_{ij} + \alpha_i\alpha_j M, \tag{19}$$

together with (1) and (7), one can get to an undrained constitutive relation for σ_{xx}:

$$\sigma_{xx} = M_{11}^u e_{xx} + M_{13}^u e_{zz} - \alpha_1 M\zeta - (\alpha_1 M\beta_e + \alpha_{d,1})\,T. \tag{20}$$

In the case of an infinite slab with symmetry for translation in the longitudinal direction, flow will only occur in transverse direction to the slab, which can be estimated via Darcy's law:

$$q_x = -\frac{k}{\gamma}\frac{\partial p}{\partial x}, \tag{21}$$

where q_x is the tranverse volumetric flux, k is the hydraulic condutivity and γ is the specific weight. The continuity equation in this case is expressed as

$$\frac{\partial\zeta}{\partial t} + \frac{\partial q_x}{\partial x} = \frac{\partial\zeta}{\partial t} - \frac{k}{\gamma}\frac{\partial^2 p}{\partial x^2} = 0. \tag{22}$$

Substituting p of eq. (7) in the above equation, one gets

$$\frac{\partial \zeta}{\partial t} - \frac{k}{\gamma} M \frac{\partial^2 \zeta}{\partial x^2} + \frac{k}{\gamma} M \alpha_1 \frac{\partial^2 e_{xx}}{\partial x^2} - \frac{k}{\gamma} M \beta_e \frac{\partial^2 T}{\partial x^2} = 0. \tag{23}$$

Here, it was invoked the Navier hypothesis, which enforces $\frac{\partial^2 e_{zz}}{\partial x^2} = 0$. The same hypothesis can be applied to eq. (20); as there are no transverse loads in the present problem, σ_{xx} vanishes, which leads to

$$\frac{\partial^2 e_{xx}}{\partial x^2} = \frac{\alpha_1 M}{M_{11}^u} \frac{\partial^2 \zeta}{\partial x^2} + \frac{\alpha_1 M \beta_e + \alpha_{d,1}}{M_{11}^u} \frac{\partial^2 T}{\partial x^2}. \tag{24}$$

By substitution of this result in eq. (23), one gets

$$\frac{\partial \zeta}{\partial t} - c_1 \frac{\partial^2 \zeta}{\partial x^2} + c_t \frac{\partial^2 T}{\partial x^2} = 0, \qquad c_1 = \frac{k\, M}{\gamma} \frac{M_{11}}{M_{11}^u}, \qquad c_t = \frac{k\, M}{\gamma} \frac{\alpha_1 \alpha_{d,1} - M_{11} \beta_e}{M_{11}^u}, \tag{25}$$

which is the equation of Biot consolidation for Mandel's problem, with adition of the last term, which takes into account thermal expansion. Substituion of ζ (eq. 16) in the above equation results in

$$\left(\frac{\partial}{\partial t} - c_1 \frac{\partial^2}{\partial x^2} \right) \left[\frac{\alpha_3 M_{11} - \alpha_1 M_{13}}{M_{11} M_{33} - M_{13}^2} \left(\sigma_{zz} + A_1 p + A_t T \right) \right] + c_t \frac{\partial^2 T}{\partial x^2} = 0. \tag{26}$$

3 Heat transfer model

In this work, for the sake of simplicity, the heat transfer within the slab will be considered only as diffusion process, i.e., the convective process due to water flow will be neglected. This approximation, which simplifies the solution in several ways, seem to be reasonable, as the hydraulic conductivity of the Carrara marble is very low, as well as its porosity. The Fourier law, which governs thermal diffusion in its usual form for an homogeneus medium in one direction (transversal), is given by

$$\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2}, \quad \text{with} \quad D = \frac{k_t}{\rho c}, \tag{27}$$

where k_t is the thermal conductivity of the medium, ρ is its density and c is its specific heat capacity. One should bear in mind that all these properties should be taken for the "composite" medium, encompassing both calcite and water properties, as well as its the influence of its geometric arrangement to the heat flow.

Figure 1 presents the temperature readings of a Carrara marble slab that was used as building envelope of Pescara Justice Court. It can be noticed that, according to Ito et al. [2], the fast temperature drawdown that took place at 14:00 was responsible to the highest value of σ_{yy} along all that day. Studies carried by Ito [6] showed that this peak value of σ_{yy} was also well approximated by a polynomial solution which was a linear funcion of time t and a quadratic function of the transverse distance from the slab face ($0 \le x \le L$). This approximation will be used in this work as an approximation of the temperature pattern in order to estimate the darcian flow and the pore pressure profile in the slab.

The polynomial solution for thermal diffusion has the form

$$T = T_0 \left[2 - \frac{4}{L^2} x^2 - 8 \frac{D}{L^2} t \right], \tag{28}$$

where T_0 is the "gauged" peak temperature. By substitution in (25), one gets

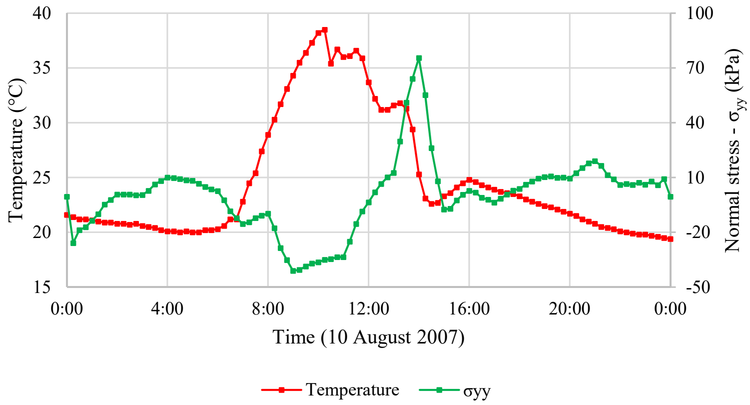

Figure 1. Temperature readings and stress estimation in the slab – adapted from Ito et al. [2]

$$\frac{\partial \zeta}{\partial t} - c_1 \frac{\partial^2 \zeta}{\partial x^2} + c_t \frac{8T_0}{L^2} \quad = \quad 0. \tag{29}$$

4 Polynomial solution for dissipation

For the same sake of simplicity, a polynomial solution is sought, bearing in mind that the "linear ramp" of temperature drawdown is causing a "steady" linear descent, in which any different initial condition has been dissipated. In this case, ζ will also have a linear behavior in time:

$$\frac{\partial \zeta}{\partial t} \quad = \quad c_{zt}. \tag{30}$$

Substituting it in eq. (29) and taking into account an approximate symmetry in $x = 0$, the solution takes the general form

$$\zeta \quad = \quad c_{zt} t - \frac{1}{c_1} \left(c_{zt} - c_t \frac{8T_0}{L^2} \right) \frac{x^2}{2} + c_{z0}, \tag{31}$$

with c_{zt} and c_{z0} being integration constants to be determined. By substitution in (16), one gets

$$\sigma_{zz} + A_1 p = \frac{M_{11} M_{33} - M_{13}^2}{\alpha_3 M_{11} - \alpha_1 M_{13}} \left[c_{zt} t - \frac{1}{c_1} \left(c_{zt} - c_t \frac{8T_0}{L^2} \right) \frac{x^2}{2} + c_{z0} \right] - A_t T. \tag{32}$$

Direct application of Navier hypothesis and symmetry for $x = 0$ in eq. (16), one gets that ζ must be independent of x, i.e., it must be just a function of t. Following the polynomial character of this solution, this can be a linear fuction

$$\sigma_{zz} + \frac{M_{11}\alpha_3 - M_{13}\alpha_1}{M_{11}} p + \frac{M_{11}\alpha_{d,3} - M_{13}\alpha_{d,1}}{M_{11}} T \quad = \quad C_0 + C_1 t. \tag{33}$$

By subtracting (33) from (32), one gets

$$\begin{aligned} A_1 p - \frac{M_{11}\alpha_3 - M_{13}\alpha_1}{M_1 1} p &= A_1 p - A_2 p \\ &= \frac{M_{11}M_{33} - M_{13}^2}{\alpha_3 M_{11} - \alpha_1 M_{13}} \left[c_{zt} t - \frac{1}{c_1}\left(c_{zt} - c_t \frac{8T_0}{L^2} \right) \frac{x^2}{2} + c_{z0} \right] + \\ & \left(\frac{M_{11}\alpha_{d,3} - M_{13}\alpha_{d,1}}{M_{11}} - A_t \right) T_0 \left[2 - \frac{4}{L^2}x^2 - \frac{8D}{L^2} t \right] - C_0 - C_1 t. \end{aligned} \quad (34)$$

The entire left hand side is a function of p, which is defines the boundary conditions of the present problem. In order to produce a boundary condition constant in time, all the time-dependent terms of the right hand side should vanish. This enforces that

$$c_{zt} = \frac{\alpha_3 M_{11} - \alpha_1 M_{13}}{M_{11}M_{33} - M_{13}^2} \left[\left(\frac{M_{11}\alpha_{d,3} - M_{13}\alpha_{d,1}}{M_{11}} - A_t \right) T_0 \frac{8D}{L^2} - C_1 \right]. \quad (35)$$

By selecting only the time-independent terms of (34), one gets

$$\begin{aligned} A_1 p - A_2 p &= \frac{M_{11}M_{33} - M_{13}^2}{\alpha_3 M_{11} - \alpha_1 M_{13}} \left[\frac{1}{c_1}\left(c_t \frac{8T_0}{L^2} - c_{zt} \right) \frac{x^2}{2} - c_{z0} \right] \\ & + \left(\frac{M_{11}\alpha_{d,3} - M_{13}\alpha_{d,1}}{M_{11}} - A_t \right) T_0 \left(2 - \frac{4}{L^2}x^2 \right). \end{aligned} \quad (36)$$

By enforcing $A_1 p - A_2 p = 0$ in $x = L/2$, one gets

$$p = \frac{1}{A_1 - A_2} \left\{ \frac{M_{11}M_{33} - M_{13}^2}{\alpha_3 M_{11} - \alpha_1 M_{13}} \left[\frac{1}{c_1}\left(\frac{L^2}{8} c_{zt} - c_t T_0 \right) \right] + \left(\frac{M_{11}\alpha_{d,3} - M_{13}\alpha_{d,1}}{M_1 1} - A_t \right) T_0 \right\} \left(1 - \frac{4}{L^2}x^2 \right). \quad (37)$$

This equation has an unknown, say c_{zt}. This can be determined by integrating eq. (33) along x interval $(-L/2, L/2)$. As the integral of σ_{xx} should vanish, substitution of (28) and (33) produces an identity involving terms which are independent and terms that are dependent of t. The dependent terms produce the identity

$$C_1 = \frac{M_{11}\alpha_{d,3} - M_{13}\alpha_{d,1}}{M_{11}} T_0 \frac{8D}{L^2}, \quad (38)$$

which may be substituted in (35) to obtain finally

$$c_{zt} = \frac{\alpha_3 M_{11} - \alpha_1 M_{13}}{M_{11}M_{33} - M_{13}^2} A_t T_0 \frac{8D}{L^2}. \quad (39)$$

5 Some quantitative results and discussion

The formulation presented in the previous section is now used in order to present the magnitude of pore pressure induced in a marble slab, for the same problem presented by Guimarães et al. [3]. The same values of $D = 1.18 \cdot 10^{-6}$ m^2/s, $L = 0.03$ m, $E = 52.4$ GPa, $\nu = 0.16$, $\beta = 1.77 \cdot 10^{-5}$, $K_s = 130$ GPa and $K_f = 2.2$ GPa are the same used in that work [3]. Together with hydraulic conductivity of an unwheatered Carrara marble estimated as [7]

$$k = \frac{K\,\gamma_w}{\mu} = \frac{10^{-19.5}\,9.8}{10^{-6}} = 3.09 \cdot 10^{-13}\text{m/s}, \tag{40}$$

where K, γ_w, μ are respectively the intrinsic permeability, water specific weight and water viscosity. The following partial results were produced:

$$\begin{aligned}
B &= 0.7775 \text{ (equation 12)} & M &= 66.13304907 \text{ GPa (equation 13)} \\
\beta_e &= 1.17 \cdot 10^{-5}\,{}^\circ\text{C}^{-1} \text{ (equation 8)} & A_1 &= 2.856349015 \text{ (equation 17)} \\
A_2 &= 0.648992266 \text{ (equation 34)} & A_t &= -59.49049361 \text{ GPa}^\circ\text{C}^{-1} \text{ (equation 5)} \\
\frac{\partial T}{\partial t} &= -0.002319321^\circ\text{C}\cdot\text{ s}^{-1} \text{ (Pescara's data)} & T_0 &= 7.370723701^\circ\text{C (Pescara's data)} \\
C_1 &= 28.45401977 \text{ GPa}\cdot\text{s}^{-1} \text{ (equation 38)} & c_{zt} &= -7.0365 \cdot 10^{-08}\,{}^\circ\text{C}^{-1}\cdot\text{ s}^{-1} \text{ (equation 39)} \\
M_{11} &= 55.80121704 \text{ GPa (Abousleiman et al. [4])} & M_{11}^u &= 98.30605219 \text{ GPa (equation 19)} \\
c_t &= -6.11516 \cdot 10^{-12}\text{ m}^2\,{}^\circ\text{C}^{-1}\text{s}^{-1} \text{ (equation 25)} & c_1 &= 1.18519 \cdot 10^{-06}\text{ m}^2\text{ s}^{-1} \text{ (equation 25)}
\end{aligned}$$

By putting all these numerical values in (37), one gets to a maximum of $p = 1.37$ MPa at $x = 0$. This result is higher than the undrained maximum calculated by Guimarães et al. [3] by an order of magnitude.

The “steady” condition of temperature fall admited here may be conservative when compared to the transients that took place in Pescara. Nevertheless, this solution, after future verifications, will be useful as an analytical solution to be use in convergence analysis of numerical implementations of the thermomechanical model presented here. Future works will also use most of the theoretical background presented here in order to develop an analytical solution for Madel’s problem with periodic thermal loads.

6 Conclusions

This paper presented the main equations of porothermomechanics, which will be used in future works to model the development of pore pressures due to thermal loads and its dissipation by transient water flow. The simple analytical solution herein will also be useful in assessing the convergence of numerical implementations.

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