Modeling and Analysis of Fractures in Concrete Beams Using DBEM and the BEMCracker2D Program

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Abstract. In its remarkable progress, Fracture Mechanics, equipped with Computational Mechanics, explained the huge lack of structural norms in the face of structural failures conceived in fragile materials, still allocated under stresses below the design limit stress. Generally, such failures result from events ranging from structural weaknesses to the occurrence of failure by the brittle fracture mechanism. In this context, this work is equipped with concepts related to Fracture Mechanics, in particular, to Linear Elastic Fracture Mechanics, LEFM, to Boundary Element Method, BEM, and to Double Boundary Element Method, DBEM, and seeks to implement and attest the BEMCRACKER2D software as an effective tool to predict the propagation of multiple critical shear cracks in simple concrete elements, with the consent of discontinuities along their interfaces. As a methodology for this finding, at first, the necessary adjustments will be made in the BEMLAB2D graphical interface so that it is possible to model cracks with several segments. Subsequently, simulations of propagation of shear cracks in plain concrete will be carried out, in which their results will be used to evaluate their respective Stress Intensity Factors in light of Integral J, as well as the direction and propagation path obtained through software, in the application of the Maximum Circumferential Stress criterion, MCS, and fatigue crack propagation, resulting in the comparison with other works already consolidated in the application of predictive analysis of similar cases.

Keywords: BEM, DBEM, Fracture, Concrete, BEMCRACKER2D.

1 Introduction

In its remarkable progress, the Linear Elastic Fracture Mechanics, MLEF, equipped with Computational Mechanics, evidenced the enormous lack of structural norms in the face of structural failures conceived in fragile materials, still allocated under stresses lower than the design limit stress. Generally, such failures result from events ranging from structural weaknesses to the occurrence of failure by the brittle fracture mechanism [1,2]. Characterized by the arbitrary arrangement of its granular constituents, concrete is a fragile material, anisotropic composite subject to singular phenomena, such as the size effect. Macroscopically, it is taken as a material resulting from the union between cement paste, aggregates and a transition zone between them, composed of a portion of material already weakened, even before its mechanical action due to the presence of numerous microfaults resulting mainly from the high accumulation of water in the region [3, 4].

As long as it is not subjected to the crack propagation stage, known as cracking, concrete can be accurately represented as an isotropic and homogeneous material with linear elastic properties. After cracking, the material continues to show linear elasticity, but successive increases in tension and displacements appear on the crack, which, in an element subjected to different loads, has its mechanical behavior managed mainly by the propagation of these microfaults, expressing non-linear responses which commonly result in local failures in the part [3,5].

Parvanova and Gospodinov [6] point out that in wall beams, also known as deep beams or shear beams, made of simple or reinforced concrete, bi-supported and not subjected to initial loads, there may be an abrupt appearance of expressive diagonal traction cracks close to the supports, that dominate the element failure process until failure. In situations of initial loading on the shear beam, the appearance of flexural cracks that propagate in mode I of opening is predicted. The simplicity of these isolated cases, together with the use of the discrete crack model and the LEFM, allows an easy analysis and determination of Stress Intensity Factors (SIFs), which does not happen in scenarios resulting from failures in which the crack opening, equipped with great sliding and friction forces from the interlocking of the aggregate on the crack faces, result in the mixed opening mode, which expresses a diagonal crack considered critical or of shear.

Such properties inherent to concrete require that the methodologies used in the study of the prediction of the cracking path in concrete elements can consider the heterogeneity of the material. There are several methodologies adopted for this type of analysis, using for this work the use of the BEMCracker2D software, which, equipped with the BEMLab2D modeling software, has shown excellent results in LEFM analysis in elements made up of homogeneous materials subjected to restrictions and/or loads using the Boundary Element Method (BEM), for cases in which cracking does not occur, and the Dual Boundary Element Method (DBEM), for cases where it does occur. Its methodology includes the incremental determination of the SIFs, associated with the calculation of the integral J, of the crack propagation direction, using the Maximum Circumferential Stress (MCS) method, and of the cracking path, by adopting the crack propagation analysis by Fatigue [7].

The preference for the software is due to the intention to investigate it and enable it to carry out prospecting and propagation predictions of multiple critical shear cracks in simple concrete elements, with acquiescence of discontinuities in the course of their interfaces and application of the numerical model of DBEM via LEFM. Thus, initially, the same problems proposed by Parvanova and Gospodinov will be considered, which presents DBEM application procedures for the analysis of propagation of multiple cracks in simple concrete elements subject to cracking in mixed mode, occurring due to bending and shear. In the work, two solutions are highlighted with the implementation of the method, which for this work, have the purpose of verifying the current behavior of the BEMCracker2D software in the face of similar problems. By these models, the SIFs, the cracking path and the deformation will be evaluated for each case.

2 Theoretical Reference

2.1 Linear Elastic Fracture Mechanics (LEFM)

This study took its first steps in the mid-1920s through the work of Griffith, being continued by Williams and Irwin, in mid-1957, and by Rice, in mid-1968. Basically, LEFM is the specialty of Fracture Mechanics responsible for the analysis of fragile materials, performed in a highly simplified, but enhanced way, repairing the limitations concerning the presence of discontinuities of the traditional concepts used in the strength of materials. It is applicable to any material, provided that its deformation is restricted to a considerably elastic region, that is, a very small plastic region in relation to the total length of the crack or the dimensions of the part, allowing the plastic phenomena to be neglected [8-11]. This approach aims to determine its behavior, its gravity and, similarly, the stress and displacement fields located near the crack tip. Occurring by the use of characterization parameters, such as the SIFs, and the energy release rate, G. It is worth mentioning that the adoption of the LEFM assumes that the stress, deformation and displacement fields can be determined by the concept of SIF near the crack tip [9-13].

2.1.1 Stress Intensity Factors (SIFs)

For LEFM, the Stress Intensity Factors (SIF), usually represented by K, are individually responsible for determining the behavior of the crack, being the only characterizer of the stress field of a structural element in the presence of a sharp crack and, by this is used as a parameter to verify the structural integrity of structural components in the presence of cracks, changing correspondingly to variations in size, geometry and loading under which the part is subjected, and its value may increase linearly-elastically until if it reaches its critical, known as Material Fracture Tenacity, K_c. And, after this point, rising continuously, but in an unstable way, occurring even without the addition of external loads, ending up in the fragile fracture of the element [1,7,14-16].

Generally associated with a crack opening mode, the parameter K is described together with specific subscripts for each mode, represented here by m. The SIFs were numerically defined by Broek [17] and manifested by Anderson [8], who states that, for a given polar coordinate axis, with the origin at the crack tip, the stress field of any structural body can be defined under situations of elastic linear cracking, explaining that the stress fields at the crack tip of the analyzed material will be provided by:

$$\lim_{r \to 0} \sigma_{ij}^{(m)} = \frac{K_m}{(2\pi r)^{1/2}} f_{ij}^{(m)}(\theta) \tag{1}$$

Being subject to the principle of linear superposition that, finally, makes it possible to obtain the singular fields of stress and deformation in the vicinity of the crack tip for the three opening modes.

2.1.2 Criteria for Propagation Direction

Currently, there are several criteria for predicting the propagation direction of a given crack, among which three are most commonly used, namely the Maximum Circumferential Voltage, MSC, the Maximum Potential Energy Release Rate, MPERR, and the Minimum Density of Deformation Energy, MDDE. These are immensely relevant criteria on numerical methods designed to predict the cracking path in structural elements, making it possible to more assertively describe the path taken by the phenomenon [18]. Based on the software used, this work will portray only about the MCS, already implemented in BEMCracker2D. This method assumes as the cracking direction that perpendicular to the maximum principal stress, $\sigma_{\theta máx}$, and normal to the maximum circumferential stress, $\sigma_{rmáx}$, defined by the Strength of Materials and normal on the planes in which the shear stress, $\tau_{r\theta}$, is zero. The stresses at the crack tip are obtained by the sum of the individual stresses for modes I and II, expressed in polar coordinates for the critical plane [18,19], by:

$$cos\left(\frac{\theta}{2}\right)\left[K_I cos^2\left(\frac{\theta}{2}\right) - \frac{3}{2}K_{II} sin\theta\right] = \sigma_\theta\sqrt{2\pi r} \tag{2}$$

$$cos\left(\frac{\theta}{2}\right)\left[K_I cos^2\left(\frac{\theta}{2}\right) + K_{II}(3cos\theta - 1)\right] = 0 \tag{3}$$

The solutions of these equations provide a trivial and a non-trivial solution, from which the values of θ will be extracted [18].

2.1.3 J Integral

Appearing in 1951, the concept of the J Integral was conceived by Eshelby and re-introduced by Rice [20] as a line integral, constituted by the rate of potential energy for a nonlinear elastic solid, which runs along the edges of the notch along of an arbitrary curvilinear contour, Γ, counterclockwise, expressing the same values for whatever path is chosen around a class of notches, juxtaposed in a two-dimensional strain field of linear or non-linear elastic materials. Subsequently, it was established by Portela [21] as an effective and accurate way of obtaining SIFs with the help of BEM and its variations, indicating its main correlations, such as:

$$J_1 = \frac{K_I^2 + K_{II}^2}{E'} \tag{4}$$

and,

$$J_2 = -\frac{2K_I K_{II}}{E'} \tag{5}$$

Where E' expresses different equations for plane stress and strain conditions, being, respectively, $E' = E$ and $E' = E/(1 - v^2)$. Given a closed contour Γ which, when subjected to the principle of superposition, shows that the sum of all the energy of the contours that compose it will be null [8,18], and that on the crack face the tension vector $T_i = d_y = 0$, which result in the relations, $J_1 = -J_2$ and $J_3 = J_4 = 0$. Briefly, Moura [19] says that the implementation of the J integral has the objective of forming two paths, one closer to the crack and the other, slightly, further away, respectively represented by Γ_1 and Γ_2, as previously mentioned. Since Γ_1, considered the most complex path of a closed path and Γ_2, the least complex, by applying the aforementioned relations on them, it is possible to obtain the value of J_1 through the calculations used to determine J_2, making unnecessary the use of more sophisticated calculations.

2.2 Boundary Element Method (BEM)

Over the years, several advances were achieved and in the mid-1980s, BEM stood out with its efficiency in evaluating the SIF and analyzing cracks in the light of the LEFM only using the discretization of the contour of the problem over the elements [22,23]. This made it possible to solve problems through the transformation of partial differential equations, descriptive of the behavior of internal and external unknowns to the problem domain, relating only to the values of its boundary and, if required, to internal points of values obtained directly along the contour. Numerically, it favored the discretization of two- and three-dimensional problems by systems of equations notably smaller than those employed in differential methods, providing formidable reductions in its modeling effort [24-26]. The BEM also presents a fully filled and asymmetric matrix of coefficients, with the solution time given by the cubic power of the total degrees of freedom, increasing proportionally to the complexity of the model [24,25]. According to Moura [19] and Gomes [25], the formulation for a general numerical procedure for boundary problems is given by:

$$c^i u^i = \sum_{j=1}^{N} \left(\int_{\Gamma_j} u^* \, \phi^T \mathrm{d}\Gamma \right) p^n - \left(\int_{\Gamma_j} p^* \, \phi^T \mathrm{d}\Gamma \right) u^n \tag{6}$$

Where Γ_j is the surface of element j and ϕ is the interpolation function of the N boundary elements. For two-dimensional situations ($i = 1{,}2$ and $j = 1{,}2$), it is assumed the presence of submatrices 2x2, h_{ij}^k and g_{ij}^k, in which the index k expresses the number of nodes in the element. The development of the formulation occurs until the form of matrix equations is reached, used successively at each node, resulting in obtaining the solution, only after applying the boundary conditions, evidencing the following system:

$$Hu = Gp \tag{7}$$

Thus, equation (7) is adequate for a given point load allocated on the boundary ξ_1 and, therefore, the values of u^* and p^*, are known, and the values of u and p, are unknown, belonging to the boundary, and the unknown c can be obtained analytically.

2.3 Formulation of the Dual Boundary Element Method (DBEM)

Presented by Oliveira [27] as a method that has simplified cracking area modeling, direct calculation of the K, parameter, reduced execution time and accurate cracking simulation, the Dual Boundary Element Method (DBEM) was developed for two-dimensional problems by Portela, Aliabadi and Rook [21], in 1992, and extended to three-dimensional problems by Mi and Aliabadi, still in the same year, solving the remeshing problems commonly present in finite element and multiregional boundary element methods because only one region is used. analysis, not requiring the generation of new meshes in an incremental analysis. Simply put, the DBEM is composed of the integral contour equations of displacements, u_i, and of tensions, t_i, applied singularly to each of the crack faces [21,22,27]. Assuming the methodologies provided by Portela et al [12] and the approach by Rodrigues [2], we have that the integral boundary equations that constitute the DBEM in a tension-free crack are, respectively:

$$c_{ij}(x')u_j(x') + CPV \int_{\Gamma_c} T_{ij}(x', x)\, u_j(x) \mathrm{d}\Gamma(x) = 0 \tag{8}$$

$$n_i(x')HPV \int_{\Gamma_c} S_{ijk}(X', x)\, u_k(x) \mathrm{d}\Gamma(x) = 0 \tag{9}$$

Finally, Γ_c characterizes the contour of a crack. Both the Cauchy and Hadamard principal value integrals are finite parts of improper integrals.

2.4 About the Software

BEMCracker2D: Originally conceived by Gomes [28], with regard to conventional BEM modeling and complemented in its incremental analysis strategy by Aliabadi [24], the BEMCracker2D program, written in the C++ programming language, is structured through the centralization of processes applied to the concepts of Object Oriented Programming (OOP). Basically, BEMCracker2D involves three processing modules: Module I (standard BEM); Module II (DBEM Without Propagation) and; Module III (With Propagation). In turn, its stress analysis occurs by the BEM. Regarding their assessments, the SIFs assessment is carried out through the Integral J, the Direction/Correction of the cracking according to the MCS criteria and the Fatigue life assessment using the Paris Law;

BEMLab2D: Responsible for pre- and post-processing, BEMLab2D is a graphical interface of the GUI type, aimed at generating and visualizing different types of two-dimensional mesh, as well as for the analysis of elastostatic problems, through the computational implementation of MATLAB [29]. Together with BEMCracker2D, BEMLab2D is based on actions defined through the user-software interface, allowing the assignment of functions through simple interaction tools, such as buttons, mouse and dialogs. When running BEMLab2D, it will display its work area, with its juxtaposed modules, having its important function of modeling and adjusting the work area for the insertion of the study, as in Fig. 1 [28].

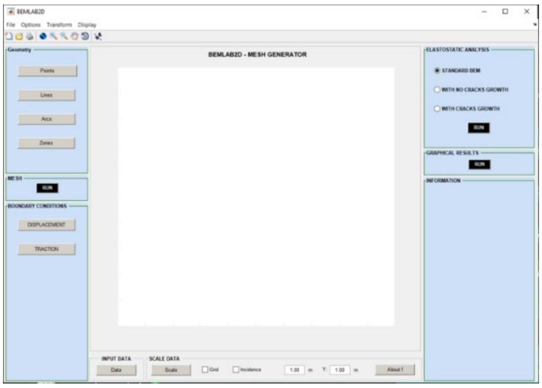

Figure 1. BEMLab2D Graphical Interface – Workspace: own font

2.5 Examples and Analysis

In order to validate the BEMCracker2D program, two mixed-mode cracking problems were simulated, as presented by Parvanova and Gospodinov [6]. For this work, they will be called Case 01 and Case 02, respectively. By means of information from the analysis models, the need for implementation was preliminarily found for case 01, given the lack of information on the express measurement units and as disparities between the methods used in the work and to those employed by BEMCracker2D. Parvanova and Gospodinov [6] applied the principle of symmetry, discretizing and analyzing only half of the beam, and representing a bending crack "frozen" of 0.5 mm.

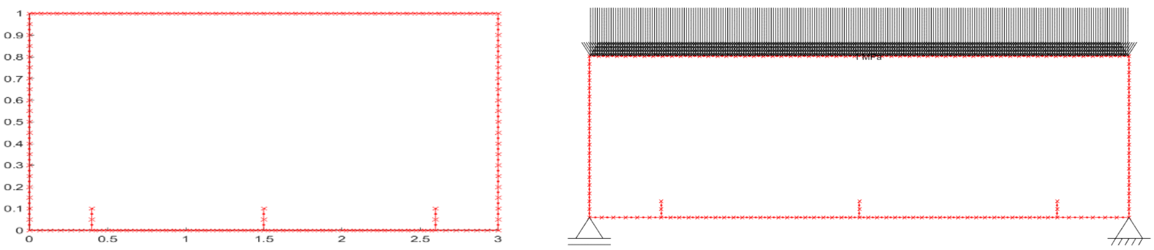

Figure 2. Case 01: (a) Geometric data; (b) Loadings and constraints: experimental data

However, in this work, the modeling of the complete beam was carried out in two moments, according to the compatibility expressed in Fig. 2. As the purpose of this work was to carry out the analysis without freezing the flexural crack, it was necessary to match its initial length with the length of diagonal tensile cracks, so that the behavior of the element before fracture could be better described. The printed distances are given in meters, as shown in Fig. 2(a), the uniformly distributed load of 1 MPa, as seen in Fig. 2(b), Young's modulus of 100 GPa and Poisson's ratio of 0.3. Cracking analysis was performed with 8 increments of 0.1m, with Paris parameters worth $C = 8.59 \cdot 10^{-14}$, $m = 3.300$ and $\text{stress ratio} = 2/3$. In the graph of Fig. 4 it's possible to see the evolutionary behavior of the SIFs by increment for both works. Emphasizing that in the reference work, only 4 increments were performed.

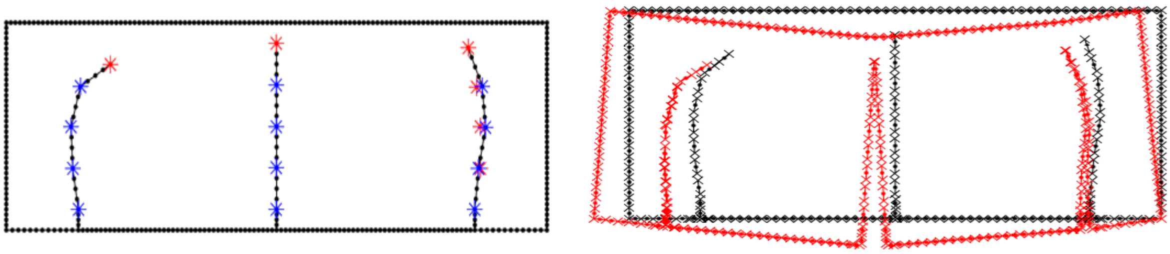

Figure 3. Case 01: (a) Cracking path (b) Deformed mesh: experimental data

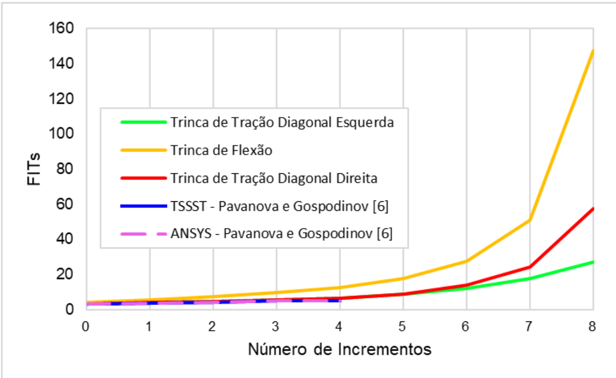

Figure 4. Case 01: Behavior of SIFs due to increase in the number of increments: experimental data

In Case 02, the modeling of the notched shear beam was implemented as in the work of Parvanova and Gospodinov [6], distinguishing themselves in the determination of SIFs, fixed in their work at the critical value $K_c = 1.65MPa \cdot m^{1/2}$ and in this work, being calculated incrementally using the BEMCracker2D program.

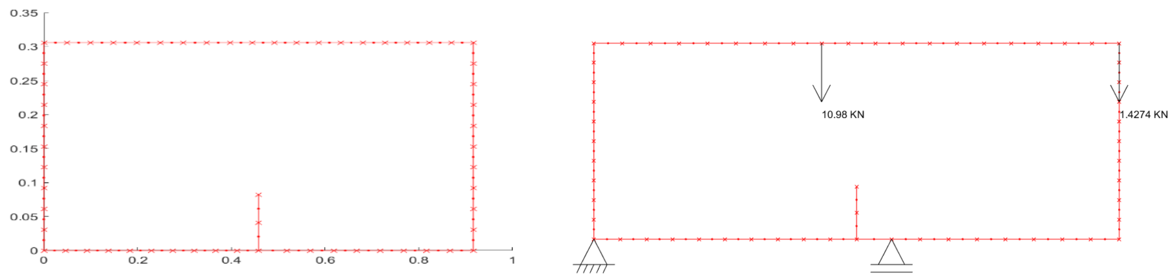

Figure 5. Case 02: (a) Geometric data; (b) Loadings and constraints: experimental data

The distances expressed in Fig. 5(a) are given in meters, with the height and total length of the beam being equal to 0.306m and 0.916m, respectively. The notch is 0.082m long and is centered on the beam. The support and the central load are 0.061m away from the center of the beam. The point loads present the value of 10.9800kN and 1.4274kN, as explained in Fig. 5(b), Young's Modulus has a value of 28.4GPa and a Poisson's ratio of 0.18. The crack propagation was simulated with 10 increments of 0.02m in length and the Paris parameters used were $C = 8.59 \cdot 10^{-14}$, $m = 3.300$ and the stress ratio $= 0.5$.

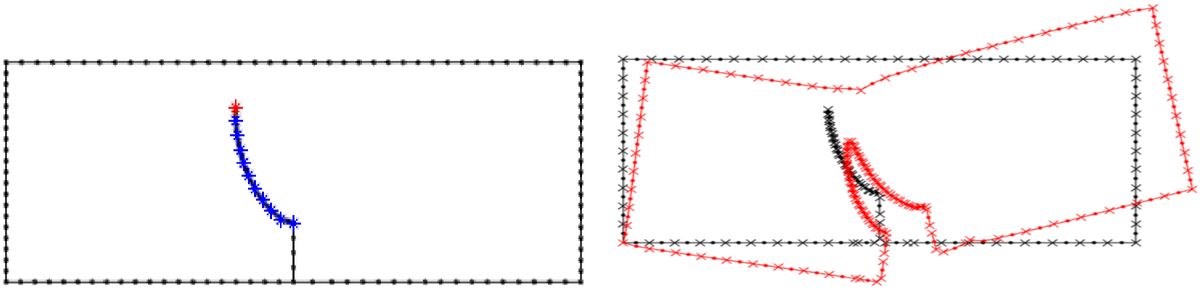

Figure 6. Case 02: (a) Cracking path (b) Deformed mesh: experimental data

3 Conclusions

Based on numerical experimentation, comparisons, analyzes and validations carried out through the application of the BEMCracker2D program, together with its graphical interface, the BEMLab2D program, to the examples presented by the work of Parvanova and Gospodinov [6], it was possible to reach the following conclusions:

- As expressed in the behavior graph of the SIFs in Fig. 4, the program showed excellent results in determining the SIFs for Case 01, with small variations resulting from the methodological distinction used by the reference article, wherein, due to its purely academic applicability, arbitrarily opted for the length and direction of their respective increments. In turn, despite include lengths close to those provided by the reference work for modeling the problem, BEMCracker2D incrementally performed the calculation and verification of the cracking direction through the MCS criterion, presenting numerically more assertive propagation paths, when compared to the manifests by the reference article;

- The program showed great precision in the calculation and graphic representation of deformed meshes and crack propagation paths for both cases, as shown in Fig. 3 and in Fig. 6, making its potential and applicability visible even in more complex models;

Therefore, it can be concluded from this work that the BEMCracker2D program, implemented by the BEMLab2D program, was able to carry out prospecting and propagation predictions of multiple critical shear cracks in simple concrete elements, with acquiescence of discontinuities along their interfaces, with the incorporation of the numerical model DBEM via LEFM.

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Declaration of Authorship. The authors confirm that they are solely responsible for the authorship of this work and that all material included herein as part of this work is owned (and authored) by the authors, or has the owners' permission to be included here.

References

[1] G. Gomes, G. Silva, G. Paiva and A. Martins, “Avaliação Numérica de SIFs em Problemas de Múltiplas Trincas”. *Iberian Latin American Congress on Computational Methods in Engineering - CILAMCE (XXXVII)*, pp. 1-6, 2016-a.
[2] W. M. Rodrigues, “Simulação de Coalescência com Múltiplas Trincas Usando o BEMCRACKER2D”. MSc dissertation University of Brasília, pp. 10-30, 33-34, 2018.
[3] J. G. Rots and J. Blaauwendraad, “Crack Models for Concrete: Discrete or Smeared? Fixed, Multi-Directional or Rotating?”. Heron, vol. 34, n. 1, pp. 7-20, 1989.
[4] M. L. Williams, “On the Stress Distribution at the Base of a Stationary Crack”. *Journal of Applied Mechanics*, pp. 113, 1957.
[5] S. G. F. Cordeiro, “Formulações do Boundary Element Method Aplicados à Análise Elástica e à Fratura Coesiva de Estruturas Compostas Planas”. MSc dissertation, University of São Paulo, pp. 142, 2015.

[6] S. Parvanova and G. Gospodinov, "A Dual Boundary Element Procedure for Analysis of Fracture in Concrete". *Journal of Theoretical and Applied Mechanics, Issue 1*, vol. 37, pp. 1, 1993.
[7] G. Gomes and A. C. O. Miranda, "Analysis of Crack Growth Problems Using the Object-Oriented Program BEMCracker2D. Frattura Ed Intregrità Strutturale", *Issue 45, The International Journal of Gruppo Italiano Frattura - IGF*, vol. XII, pp. 67-69, 2018.
[8] T. L. Anderson, "Fracture Mechanics: Fundamentals and Applications". *CRC Press - Taylor & Francis*, Third edition, pp. 1-10, 16-17, 42-45, 103, 156-158, 219-243, 249-250, 2005.
[9] Z. Bazant and J. Planas, "Fracture and Size Effect in Concrete and Other Quasibrittle Materials". *CRC Press LLC*, pp. 1-10, 23, 45-47, 257, 1998.
[10] P. Kumar, "Elements of Fracture Mechanics". *McGraw Hill, New Delhi*, pp. 1-10, 35-36, 45-47, 69-71, 2009.
[11] J. A. H. Medina, "Avaliação de Previsões e Fratura Elastoplástica". PhD thesis, Pontifical Catholic University of Rio de Janeiro, pp. 19, 26-33, 57-60, 2014.
[12] A. Portela, M. H. Aliabadi and D. P. Rooke, "Dual Boundary Element Incremental Analysis of Crack Propagation". *Computers & Structures*, vol. 46, Number 2, pp. 239, 1993.
[13] E. N. Serenza, O. A. G. Suarez and R. Rossi, "A Study About SIF Estimation Using XFEM". *Iberian Latin American Congress on Computational Methods in Engineering - CILAMCE (XLI)*, pp. 1-2, 2020.
[14] J. M. Barsom and S. T. Rolfe, "Fracture and Fatigue Control in Structures – Applications of Fracture Mechanics". *American Society for Testing and Materials - ASTM,* Third edition, pp. 30-35, 1999.
[15] H. Carasek, J. Ferrareto, I. Beja and J. Ceratti, "Materiais de Construção Civil e Princípios de Ciência e Engenharia dos Materiais – Introdução, Ciência dos Materiais, Rochas e Solos, Aglomerantes Minerais: Mecânica Fratura e Fraturamento do Concreto". *IBRACON*, terceira edição, vol. 1, pp. 5-6, 11-12, 18-19, 2018.
[16] N. E. Dowling, "Mechanical Behavior of Materials". *Pearson Education Limited*, fourth edition, pp. 339, 344, 350, 393-394, 2013.
[17] D. Broek, "Elementary Engineering Fracture Mechanics". *Martinus Nijhoff Publisers*, pp. 6-13, 24-48, 1984.
[18] A. C. O. Miranda, "Propagação de Trincas por Fadiga em Geometrias 2D Complexas Sob Cargas Cíclicas Variáveis". PhD thesis, Pontifical Catholic University of Rio de Janeiro, pp. 1, 8-9, 17-18, 34-39, 2003.
[19] D. C. Moura, "Caminho de Propagação de Trinca: Modelagem Automática e Comparação de Diferentes Critérios com o Programa BEMCracker2D". MSc dissertation, University of Brasília, pp. 17-18, 27-32, 2019.
[20] J. R. Rice, "A Path Independent Integral and the Approximate Analysis of Strain Concentration by Notches and Cracks". *Journal of Applied Mechanics*, pp. 2-12, 1968.
[21] A. Portela, M. H. Aliabadi and D. P. Rooke, "The Dual Boundary Element Method: Effective Implementation for Crack Problems". *International Journal for Numerical Methods in Engineering, Issue 6*, vol. 33, pp. 1270-1272, 1278-1280, 1992.
[22] M. H. Aliabadi and C. A. Brebbia, "Boundary Element Formulations in Fracture Mechanics: A Review". *Transactions on Engineering Sciences*, vol. 13, pp. 3-5, 1996.
[23] X. W Gao, "The Radial Integration Method for Evaluation of Domain Integrals with Boundary-Only Discretization". *Elsevier Science, Engineering Analysis with Boundary Elements, Issue 26, The International Journal for Advances in Boundary Element and Other Mesh Reduction Methods*, pp. 905-906, 2002.
[24] M. H. Aliabadi, "The Boundary Element Method: Applications in Solids and Structures". *Wiley - John Wiley & Sons*, vol. 2, pp. viii, 1, 4-5, 2002.
[25] G. Gomes, "Aplicação dos Métodos de Elementos de Contorno e Reciprocidade Dual em Problemas de Plasticidade 2D Orientada à Objeto". PhD thesis, University of Brasília, pp. 2-5, 21, 27, 2006.
[26] A. K. L. Kzam and H. B. Coda, "Tratamento das Integrais Singulares do Boundary Element Method". *Cadernos de Engenharia de Estruturas*, vol. 12, 18p, 2010.
[27] T. A. A. Oliveira, "Análise Multiscala Via Dual Boundary Element Method de Fuselagem de Aeronave". MSc dissertation, University of Brasília, pp. 34-36p, 2019.
[28] G. Gomes, and Á. M. Delgado Neto, "Modelling and 2D Cracks View Using Dual Boundary Integral Equation". *Iberian Latin American Congress on Computational Methods in Engineering – CILAMCE (XXXVII)*, pp. 6-9, 2016-b.
[29] A. M. Delgado Neto, "BEMLAB2D: Interface Gráfica de Modelagem, Visualização e Análise com Elementos de Contorno – Uma Aplicação em Problemas Elastostáticos". MSc dissertation, University of Brasília, pp. 34, 36-63, 2017.