XLIII CILAMCE
21-25, November, 2022
FOZ DO IGUAÇU - PR - BRAZIL

Sensor Placement Optimization for Numerical Model Reduction using Genetic Algorithm

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Abstract. Numerical reduction techniques are crucial for numerical and experimental model compatibility. Besides reducing computational costs, finite element (FE) models generally have significant number of degrees of freedom (DOFs) when compared to experimental models. Therefore, sensor placement techniques based on effective independence (EI), condition number of the modal matrix (CN) and the sum of the off-diagonal terms of the modal assurance criteria (off-MAC) matrix aim to provide optimal sensors position setup for future accurate experimental tests. Thus, this work presents a comparative study among the mentioned sensor placement techniques for selecting the candidate numerical DOFs to reduce the free-free beam FE model through the Guyan technique. In addition, it is applied genetic optimization algorithm (GA) under CN and MAC techniques to reach optimal solutions. A sensitivity analysis of the optimal responses from CN and MAC was held, along with an F-test, which ranked the relevant DOFs for sensor placement. The results showed that root-mean-square-error (RMSE) between the reduced FE and full FE models was less than 5%. MAC values were above 0.86. Finally, it was identified that the three methods need a DOF's selection of spatial constraints to circumvent possible problems of the modes poor spatial resolution of the reduced FE model.

Keywords: Condition number, Effective Independence, Guyan technique, MAC, Sensor placement

1 Introduction

In complex engineering systems, positioning sensor is crucial for executing experimental tests, allowing to obtain accurate results in the modal parameters' investigation. However, due to the high costs of installing these sensors and the inaccessibility of certain structural locations, this task may be subject to inaccuracies in the spatial resolution of the vibration modes. Therefore, due to these problems, it is necessary to develop finite element (FE) models that allow modelling of the unmeasured structural locations and providing an adequate spatial resolution of the vibration modes.

However, FE models generally have a significant number of degrees of freedom (DOFs) compared to experimental models, making the correspondence between them unfeasible. Several techniques are found in the literature to perform the reduction-expansion of FE models to carry out this correspondence. One of the techniques is the Guyan Reduction technique [1], which reduces the mass and stiffness matrices of a full FE model. Thus, applying this technique, a new reduced FE model of a particular system under study is assembled to represent only the DOFs of interest. This technique is used in numerous commercial numerical analysis packages. However, it provides higher eigenvalues than the initial FE model. One of the mentioned causes is related to the intrinsically dependence on the candidate DOFs of interest and the investigated modal shapes [2].

The model reduction must, however, be made judiciously, as the discarded DOFs during the process can significantly contribute to the system response [3]. Therefore, methods of sensors selecting and positioning, such as effective independence (EI) [5], the condition number of the modal matrix (CN) [4] and the sum of the off-diagonal terms of the modal assurance criteria matrix (off-MAC) [9] were developed to provide precision and accuracy of measurements and support the analyst in the decision-making of the sensors locations and installation.

However, it is not possible to estate that the selected proposed DOFs by the mentioned techniques are an optimal configuration [6],[7]. Thus, the techniques based on the CN and the off-MAC present the need to apply some optimization algorithm, producing a possible number of optimal solutions. In the literature, several works deal with applying optimization algorithms associated with these sensor positioning methods [8], [9], [10]. Therefore, due to the range of possible optimal solutions, it is crucial to conduct a sensitivity analysis of the responses concerning the selected numerical candidate DOFs. This task aims to evaluate the reduced FE models' precision and accuracy assembled by these solutions and to identify the DOFs' relevance in the final reduced FE model.

Therefore, this work aims to present a comparative study of the mentioned sensor placement techniques to reach the best solution to assemble the free-free beam reduced FE model and to define sensors' locations along the beam for future experimental tests. Moreover, sensitivity analysis of the obtained optimal responses from applying the GA under CN and MAC sensor placement techniques was performed, identifying which DOFs are relevant to the reduction of the FE model using the Guyan technique. Also, this work aims to demonstrate these techniques deficiency related to the selection of candidate numerical DOFs under the point of view of possible vibration modes' spatial resolution problems from the assembled reduced FE model.

This work describes in section 2 the theoretical aspects related to the Guyan reduction technique of the numerical models. In addition, CN, EI and off-MAC sensor placement techniques are described. In section 3, the optimization proposal procedures are outlined. Section 4 presents the case study, results, and discussions. Finally, section 5 presents the conclusion.

2 Theoretical aspects

2.1 Guyan Reduction Technique

The static equations of equilibrium,

$$\boldsymbol{Kx} = \boldsymbol{f}, \tag{1}$$

was used in the derivation of Guyan condensation, where $\mathbf{K} \in \mathbb{R}^{n \times n}$ is the stiffness matrix of the full FE model. According to [1], we can assume that the full model total degrees of freedom are categorized as the *active* or *primary* DOFs and the deleted DOFs, the *secondary* ones. They are simply referred to as the active and the deleted and are indicated by *a* and *d*, respectively. With this arrangement, the static eq. (1) may be partitioned as:

$$\begin{bmatrix} \boldsymbol{K}_{aa} & \boldsymbol{K}_{ad} \\ \boldsymbol{K}_{da} & \boldsymbol{K}_{dd} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x}_a \\ \boldsymbol{x}_d \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{f}_a \\ \boldsymbol{f}_d \end{Bmatrix}, \tag{2}$$

which $\boldsymbol{x}_d \in \mathbb{R}^d$ is the displacement vector corresponding to the deleted DOFs, which are to be condensed, and $\boldsymbol{x}_a \in \mathbb{R}^a$ is the vector corresponding to the active DOFs, which are to be retained.

The displacements at the deleted DOFs consist of two parts due to the linearity of this model. After some mathematical manipulations and assuming $\boldsymbol{f}_d = \boldsymbol{0}$, it can establish the following relation.

$$\boldsymbol{x}_d = \boldsymbol{R}_G \boldsymbol{x}_a, \tag{3}$$

where $\mathbf{R}_G \in \mathbb{R}^{d \times a}$ is called the condensation matrix and is defined as:

$$\boldsymbol{R}_G = -\boldsymbol{K}_{dd}^{-1} \boldsymbol{K}_{da}. \tag{4}$$

Eq. (3) establishes the displacements relations between the active and the deleted DOFs. The corresponding condensation matrix is a load-independent matrix because the external forces at the deleted DOFs were ignored in the derivation. This condensation method was first proposed by [2] and is usually referred to as Guyan condensation.

Therefore, the Guyan condensation matrix can determine the transformation matrix $\mathbf{T} \in \mathbb{R}^{n \times a}$ that allows the reducing of a full FE model as follows:

$$\boldsymbol{M}_r = \boldsymbol{T}^{\mathsf{T}}\boldsymbol{M}\boldsymbol{T} \therefore \boldsymbol{K}_r = \boldsymbol{T}^{\mathsf{T}}\boldsymbol{K}\boldsymbol{T} \therefore \boldsymbol{f}_r = \boldsymbol{T}^{\mathsf{T}}\boldsymbol{f} \rightarrow \boldsymbol{T} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{R}_G \end{bmatrix}, \tag{5}$$

which, $\boldsymbol{M} \in \mathbb{R}^{n \times n}$ is the mass matrix of the full FE model and $\boldsymbol{I} \in \mathbb{R}^{a \times a}$ is the identity matrix. Therefore, the reduced FE model can be represented accordingly as follow.

$$\boldsymbol{M}_r\ddot{\boldsymbol{x}}_a(t) + \boldsymbol{K}_r\boldsymbol{x}_a(t) = \boldsymbol{f}_r(t) \tag{6}$$

2.2 Sensor Placement Techniques

Condition Number of the Modal Matrix

The condition number for the inversion of a matrix measures the sensitivity solution of a system of linear equations to errors in the data. It gives an indication of the accuracy of ~~the~~ results from matrix inversion and the linear equation solution. A matrix with a low condition number is said to be well-conditioned for its inversion. Otherwise, a matrix with a high-condition number is said to be ill-conditioned for its inversion. According to [3], the importance of the modal matrix's inversion is in studying a dynamic system's stability. Therefore, the precise study of sensors placement may conduct to a good dynamic stability.

Effective Independence

This method aims to select sensor locations that make the mode shapes of interest as linearly independent as possible while retaining as much information as possible about the selected modal responses in the measurement data [4]. Thus, the core idea is to remove all DOFs that cannot be measured or that do not have significance in the full FE model for the modes of interest, allowing for the reduced FE model assembly. The so-called *effective independence* (EI) is based on the *Fisher Information Matrix*, $\boldsymbol{A}_{FIM}$, given by

$$\boldsymbol{A}_{FIM} = \boldsymbol{\Phi}_m^{\mathsf{T}}\, \boldsymbol{\Phi}_m. \tag{7}$$

Therefore, EI is described as follows:

$$\boldsymbol{E} = \boldsymbol{\Phi}_m\, \boldsymbol{A}_{FIM}\boldsymbol{\Phi}_m^{-1}, \tag{8}$$

where $\boldsymbol{\Phi}_m \in \mathbb{R}^{m \times m}$ is the numerical modal matriz partitioned at *m* candidate numerical DOFs and **E** is an independent matrix with the property that its trace equals its rank. Therefore, in the matrix matrix **E** , the terms contained in the diagonal represent the contribution of each possible candidate DOF to the reduced FE model (or measured location) to the rank of **E** and, hence, to the independence of the chosen modes.

The selection procedure consists of studying the elements of the diagonal of **E** iteratively. Since the smallest element relates to the DOF that contributes the least to the independence of the modes of interest, this DOF is removed. Then, the matrix **E** is recomputed, and the process is repeated until the remaining amount of DOFs is equal to the number of available sensors for performing the experimental tests.

Modal Assurance Criteria (MAC)

Modal Assurance Criteria (MAC) serves as a quality assurance indicator for modal vectors that are estimated from an experimental or numerical model [5]. MAC values range from 0 to 1, which 0 indicates inconsistency or orthogonality between eigenvectors and 1 indicates perfect consistency (differing only by a scale factor). The MAC between *i*th and *j*th modes is given by:

$$MAC_{i,j} = \frac{|\boldsymbol{\Phi}_{m,i}^{\mathsf{T}}\, \boldsymbol{\Phi}_{m,j}|^2}{(\boldsymbol{\Phi}_{m,j}^{\mathsf{T}}\, \boldsymbol{\Phi}_{m,j}) \cdot (\boldsymbol{\Phi}_{m,i}^{\mathsf{T}}\, \boldsymbol{\Phi}_{m,i})} \therefore i,j = 1,2,3,\dots,m \tag{9}$$

However, as a sensor placements method, the own FE modes are compared to each other, creating the *auto-MAC* matrix. In this case, the diagonal entries are unity; meanwhile, the off-diagonal terms are not zero because the eigenvectors are orthogonal concerning the mass and stiffness matrices, and the MAC includes no such weightings. The core idea of this technique is to select candidate DOFs that provide the minor sum of the off-diagonal terms of the MAC matrix (off-MAC technique). According to [6], a significant sum of the off-diagonal terms represents a more dependence between the interest eigenvectors.

3 Proposed Optimization Procedures

For the present work, the genetic algorithm approach was applied to identify the best solutions to be presented by the sensor positioning techniques using the CN and off-MAC of the modal matrix partitioned by the candidate DOFs. Eqs. (10) and (11) present the formulation of the proposed optimization problem for the mentioned CN and off-MAC sensor placement techniques, respectively.

$$\min \|\boldsymbol{\Phi}_m(a_k)\| \|\boldsymbol{\Phi}_m(a_k)\|^{-1} \qquad (10)$$

with: $\boldsymbol{\Phi}_m(a_k) \in \mathbb{R}^{m \times m}$

$$\min \sum_{i,j=1}^{m} MAC\left(\boldsymbol{\Phi}_m^{i,j}(a_k), \boldsymbol{\Phi}_m^{i,j}(a_k)\right) \therefore i \neq j \qquad (11)$$

Subject to: $a_k = [0,1] \in \mathbb{N}$

$\sum_{k=1}^{p} a_k = \mathrm{m}$

where:

m: number of available sensors (and modes of interest)

p: number of full FE model nodes

a_k: kth candidate numerical DOF → $\mathrm{k} = 1,2,3, \ldots, \mathrm{p}$

$\boldsymbol{\Phi}_m(a_k)$: Modal matrix partitioned at m candidate numerical DOFs of the full FE model

4 Results and Discussion

4.1 Case Study: Numerical Free-Free Beam Model

The case study in this paper consists of a numerical rectangular free-free aluminum beam model. The FE model was assembled with 25 nodes and 24 Timoshenko beam elements, and each node has only two DOFs (z-axis translation and x-axis rotation), totaling 50 DOFs. The objective of this study is to investigate its z-axis FE modal parameters before performing the respective experimental tests. Thus, it was assumed that nine sensors were available to perform the experimental setup. Therefore, the mentioned sensor placement techniques were applied to investigate the best possible installation locations along the beam. Fig. 1 illustrates the beam's FE model.

Figure 1 - Free-Free beam FE model

4.2 Sensitivity Analysis

Due to the application of GA in sensor placement techniques based on CN and off-MAC, numerous optimal solutions for candidate DOFs will be obtained. So, it is crucial to conduct a sensitivity analysis to identify the pattern of responses obtained by the techniques. Similar procedures are applied to the generation of random numbers [11]; the ideal number of runs is defined from the initial 20 different optimal solutions in the GA processing. The descriptive statistics analysis defined that the ideal number of GA processing's for CN and off-MAC is 42 and 53, respectively, using a 95% confidence interval.

4.3 Obtained Results

From the different optimal solutions obtained from CN and off-MAC techniques, the relevance analysis of the candidate DOFs' selection was conducted through the application of the F-Test. This analysis aims to identify which candidate DOFs are relevant in all optimal solutions. Fig. 2(a) presents the number of occurrences and relevance classification of each candidate numerical DOFs in all CN optimal results. Similarly, it can be seen the same for all off-MAC optimal results in Fig. 2(b).

For the CN technique, the 12th solution was considered the best among the 42 GA's runs, as it presented the smallest RMSE between reduced and full FE model natural frequencies of 3.27%, being defined by the selection of candidate DOFs 3, 5, 8, 10, 12, 15, 20, 22 and 25. Fig. 3(a) illustrates its best solution for the selected candidate numerical DOFs (in black) and the most relevant identified DOFs (in blue). Meanwhile, for the off-MAC technique, the 24th solution was considered the best among the 53 GA's runs as it presented the smallest RMSE between reduced and full FE model natural frequencies of 3.70%, being defined by the selection of candidate DOFs 3, 6, 9, 11, 14, 17, 19, 22 and 24. Fig. 3(b) illustrates the best solution for selected candidate numerical DOFs and the most relevant identified DOFs. As for the EI technique, as it is iterative and presents a single sensor positioning solution, the obtained solution regarding the selected candidate numerical DOFs is shown in Figure 3(c).

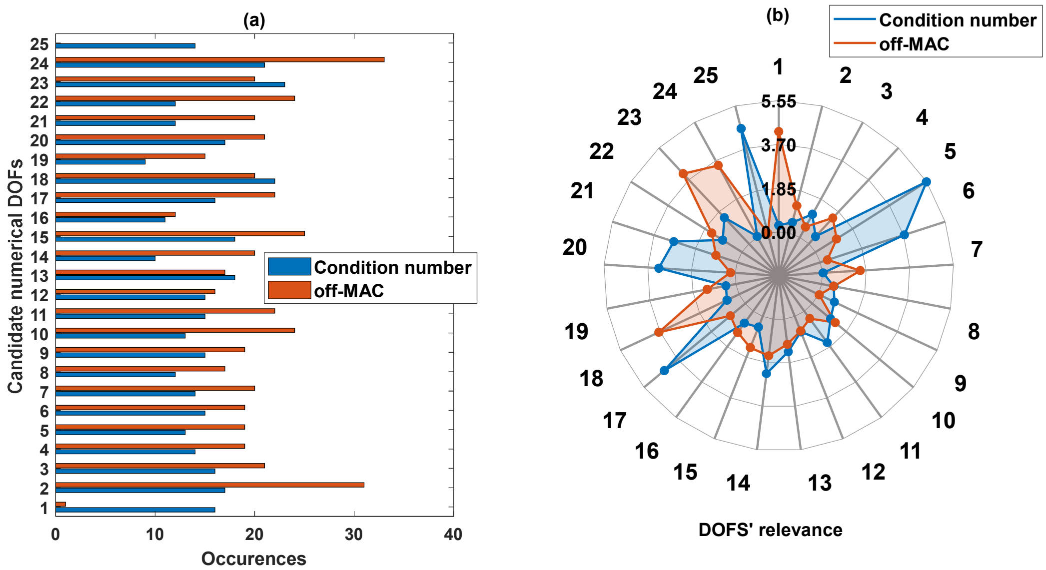

Figure 2 – (a): Number of occurrences of each candidate DOF in all optimal solutions by CN and off-MAC technique. (b): relevance of each candidate DOF in all optimal solutions by CN and off-MAC technique.

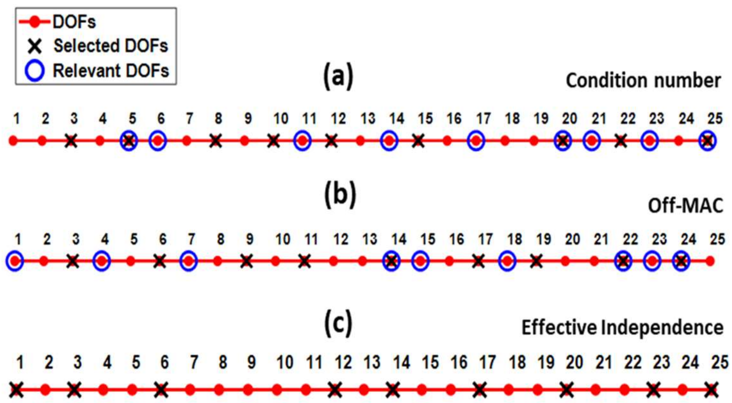

Figure 3 – (a): Best solution of selected candidate numerical DOFs by CN technique. (b): Best solution of selected candidate numerical DOFs by off-MAC technique. (c): Presented solution by EI technique.

4.4 Discussion

The precision and accuracy of the reduced models were compared using their respective best solutions and the most relevant DOFs. Table 1 presents the mean error, the variance and the RMSE values of the difference between the frequencies of the reduced and the full FE model using the best solution and the most relevant DOFs. These results are also presented for the reduced FE model by the EI technique's presented solution.

Table 1 - Comparison between reduced models from the best solutions and relevant DOFs of CN and MAC.

Technique	Solution	Error(Freq. Diff)	Variance (Freq.Diff)	RMSE (Freq.Diff%)
CN	Best solution	2,15	6,78	3,27
	Most relevant DOFs	3,57	9,13	4,57
Off-MAC	Best solution	2,80	**5,50**	3,70
	Most relevant DOFs	8,00	62,79	10,95
EI	Presented solution	**1,90**	6,22	**3,03**

Despite identifying the most relevant DOFs in each approach, these do not necessarily contribute to the determination of a global optimum [6]. Therefore, identifying these DOFs, allows the analyst to have a notion that such DOFs better divide the dataset of optimal solutions presented by the GA.

Fig. 4 (a) shows the differences between the natural frequencies obtained for the reduced models elaborated using the selected mentioned techniques DOFs and the complete beam model. Fig. 4 (b) describes the MAC values obtained in comparing the vibration modes of the reduced model and the full model. It must be noted that the natural frequency differences tend to be higher for the higher modes [1].

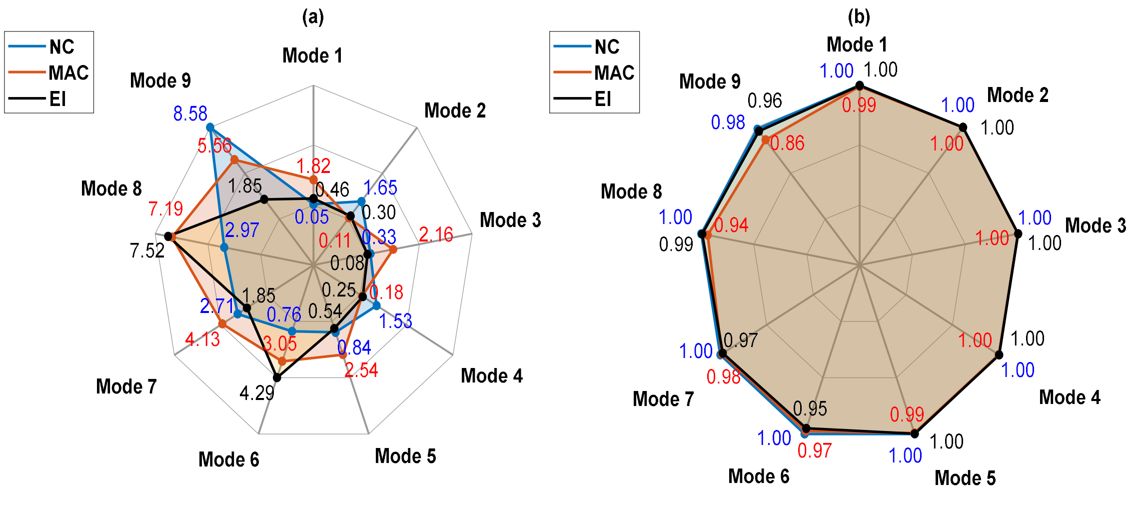

Figure 4 - (a) Frequency difference between the reduced model and the full model in [%]. (b) MAC values between reduced and full model modes.

Comparing the reduced by the best solutions models obtained by the CN, off-MAC, and EI, they returned RMSE values below 5%. Therefore, the EI solution has the lowest RMSE value among the three methods. The three methods mean error values also indicate that the EI has the lowest value. On the other hand, MAC has the lowest variance.

Figure 3 illustrates that there were significant spacings between the candidate DOFs selected by the presented techniques, and, although these techniques are well established in the literature, none of them presents any spatial consideration of the analyzed problem. Therefore, if there is a need to estimate or reconstruct signals at unmeasured locations in the structure, depending on the selected candidate DOFs, it will not be possible to obtain accurate results. Furthermore, the experimental modes may present problems of spatial resolution due to the discretization of specific locations in detriment of locations with sparse sensors positioning.

5 Conclusion

This paper compared the best sensor placement solutions for a free-free beam FE model applying the CN and off-MAC under GA and Effective Independence techniques. A subsequent model reduction was applied using the Guyan reduction technique using the obtained candidate numerical DOFs solutions.

The chosen best solutions were those that obtained the lowest values of RMSE between the natural frequencies of the full model and the reduced ones. Performing a sensitivity analysis of all optimal solutions from CN and off-MAC techniques, it was possible to identify which DOFs are relevant for the FE model reduction process. Therefore, the CN and off-MAC best solutions were compared with solutions composed of only the most relevant DOFs. Therefore, the most relevant DOFs will not necessarily be the best, but they need instrumenting attention because these tend to divide better the database of all possible candidate numerical DOFs solutions for FE model reduction.

Regarding the natural frequencies, it is noted that, due to the Guyan reduction technique, the error difference increases with the mode increase. The frequencies tend to be overestimated for higher modes from the reduced FE model. The reduced FE model by EI solution proved to have the best performance than the reduced models by the CN and off-MAC best solutions (Table 1). Nevertheless, all the solutions provided by these techniques show that sensor distribution along the beam may cause future problems concerning the spatial resolution of the experimental vibration modes. This issue can cause inaccuracies in estimating vibration response at unmeasured structural locations.

With this, this work concludes that it is necessary to insert sensor spatial position constraints in these techniques to improve the precision and accuracy of experimental tests and the prediction of responses in non-instrumented locations.

Acknowledgements. This research is supported by Laboratory of Dynamic Tests and Vibrations Analysis (LEDAV) of Federal University of Rio de Janeiro (UFRJ).

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