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A boundary element formulation for two-step thermoelastic analysis without internal cells

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Abstract. Uncoupled (two-step) thermoelastic analysis is addressed by a new boundary element method formulation that eliminates the use of internal cells. The first step is a steady-state thermal analysis. The second step is a mechanical analysis, which uses the temperature field obtained in the first step as part of the applied loads. Domain integrals in temperature, displacement and stress boundary integral equations are transformed to boundary by the radial integration method. The radial integration method is a simple and powerful method based on a pure mathematical treatment, which transforms any domain integral into a boundary and a radial ones. The radial integral is independent of geometry and no discretization is necessary for its evaluation, while the boundary integral can be solved using the existing boundary mesh. Since temperature has been evaluated only at discrete points, namely the boundary nodes and a set of internal points, the moving least square procedure is used to calculate temperature at numerical integration points during evaluation of radial integrations of the second step. The moving least square is a technique usually adopted to generate shape functions in Meshfree methods. Two representative examples are presented to demonstrate the accuracy and robustness of the proposed formulation.

Keywords: Boundary Element Method, Cell-less thermoelastic analysis, Radial Integration Method, Moving Least Square Interpolation

1 Introduction

One of the main advantages of the Boundary Element Method (BEM) when compared to other numerical methods, mostly the domain methods such as the Finite Element Method, is its capability to solve a problem with only boundary discretization. However, depending on the problem, the resultant integral equations include domain integrals, requiring additional internal discretization techniques. Examples are the presence of internal sources for potential problems, body forces or temperature fields for elastic problems and energy dissipation phenomena for inelastic problems.

The most direct idea to treat such integrals is the cell-integration scheme, which can be found in most BEM textbooks, for example, [1, 2]. Although the use of cell-discretization leads to accurate results, in a computational geometry point-of-view, the remarkable advantage of boundary-only discretization is lost. Thus, many methods with the objective to transform the domain integrals into boundary ones, avoiding the domain division into cells, have been developed. The most known are the Galerkin tensor procedure [3], the Dual Reciprocity Method (DRM) [4, 5], the Multiple Reciprocity Method (MRM) [6] and the Radial Integration Method (RIM) [7].

Particularly, the RIM is a simple and powerful method based on a pure mathematical treatment, which transforms any domain integral into a boundary and a radial ones. The radial integral is independent of geometry and no discretization is necessary for its evaluation, while the boundary integral can be solved using the existing boundary mesh. Thus, RIM can treat different types of domain integrals that appear in the same integral equation in a unified way without using particular solutions or higher order fundamental solutions. Furthermore, the technique can also removes a variety of singularities appearing in the domain integrals.

In this paper, a new cell-less BEM formulation is presented for uncoupled thermoelastic analysis. The firrst step is a pure thermal analysis in which internal heat source terms are directly transformed to boundary by RIM. Temperature is evaluated (and stored) for each boundary and internal point. The second step is a thermoelastic analysis, using the temperature field obtained in the first step. Domain integrals in displacement and stress boundary integral equations, involving initial stresses associated to thermal strains, are also directly transformed to boundary by RIM, following the procedure described by Gao [8]. Since temperature has been evaluated only at discrete points, the moving least square procedure [9, 10] is employed for its evaluation on each radial numerical integration point.

2 Boundary integral equations

The uncoupled thermoelastic problem addressed here is a two-step analysis: firstly a steady state heat transfer analysis is performed to evaluate the temperature field along the physical domain and then, using these results, a linear elastic analysis is done. In this way, the relevant Boundary Integral Equations (BIE) are the steady state heat transfer BIE, given by

$$c(\boldsymbol{\xi})\theta(\boldsymbol{\xi}) = \int_\Gamma q^*(\boldsymbol{\xi}, \mathbf{x})\theta(\mathbf{x})\, d\Gamma - \int_\Gamma \theta^*(\boldsymbol{\xi}, \mathbf{x})q(\mathbf{x})\, d\Gamma + \int_\Omega \theta^*(\boldsymbol{\xi}, \mathbf{x})b(\mathbf{x})\, d\Omega \tag{1}$$

for the first (thermal) step and the displacement and internal stress BIEs, both including an initial stress term, σ^o_{ij}, associated to temperature change, given respectively by

$$c_{ij}(\boldsymbol{\xi})u_j(\boldsymbol{\xi}) = \int_\Gamma u^*_{ij}(\boldsymbol{\xi}, \mathbf{x})t_j(\mathbf{x})d\Gamma - \int_\Gamma t^*_{ij}(\boldsymbol{\xi}, \mathbf{x})u_j(\mathbf{x})d\Gamma + \int_\Omega \epsilon^*_{ijk}(\boldsymbol{\xi}, \mathbf{x})\sigma^o_{jk}(\mathbf{x})d\Omega \tag{2}$$

$$\sigma_{ij}(\boldsymbol{\xi}) = \int_\Gamma u^*_{ijk}(\boldsymbol{\xi}, \mathbf{x})t_k(\mathbf{x})d\Gamma - \int_\Gamma t^*_{ijk}(\boldsymbol{\xi}, \mathbf{x})u_k(\mathbf{x})d\Gamma + \int_\Omega \epsilon^*_{ijkl}(\boldsymbol{\xi}, \mathbf{x})\sigma^o_{kl}(\mathbf{x})d\Omega + F_{ijkl}\sigma^o_{kl}(\boldsymbol{\xi}) \tag{3}$$

for the second (mechanical) step.

In above equations, Ω and Γ refer respectively to the problem's domain and its boundary. In eq. (1), θ is the temperature change at a given point, q is the boundary heat flux and b is the internal heat source. In eqs. (2) and (3), u_i is the displacement field, t_i are boundary tractions and σ_{ij} is the Cauchy stress tensor. Terms $\theta^*(\boldsymbol{\xi}, \mathbf{x})$ and $q^*(\boldsymbol{\xi}, \mathbf{x})$ are, respectively, the temperature and heat flux of a steady state heat transfer fundamental problem of a punctual heat source on $\boldsymbol{\xi}$ of an infinity domain, while $u^*_{ij}(\boldsymbol{\xi}, \mathbf{x})$, $t^*_{ij}(\boldsymbol{\xi}, \mathbf{x})$, $u^*_{ijk}(\boldsymbol{\xi}, \mathbf{x})$, $t^*_{ijk}(\boldsymbol{\xi}, \mathbf{x})$ and $\epsilon^*_{ijkl}(\boldsymbol{\xi}, \mathbf{x})$ are tensors associated to the Kelvin fundamental solution of a linear elastic problem over isotropic media. Explicit expressions for these terms can be found in mostly BEM textbooks, such as [2]. All of them are functions of the distance between the source point, $\boldsymbol{\xi}$ and the field point, $\mathbf{x}$, given by

$$r \equiv r(\boldsymbol{\xi}, \mathbf{x}), \quad r_i = x_i - \xi_i, \quad r = (r_i r_i)^{1/2} \tag{4}$$

Free terms $c(\boldsymbol{\xi})$ and $c_{ij}(\boldsymbol{\xi})$ are functions of source point location related to boundary. For internal points, its values are, respectively, 1 and δ_{ij} (the Kronecker delta), while for smooth boundary points, its values are $1/2$ and $\delta_{ij}/2$. Moreover, explicit expressions for the free term F_{ijkl} can be found, for example, in [1].

Boundary stress have been evaluated in this work by the well-known traction recovery technique.

Finally, the initial stress can be obtained from the temperature change through the following expression:

$$\sigma^o_{ij} = \tilde{\lambda}\delta_{ij}\theta, \qquad \tilde{\lambda} = \begin{cases} \dfrac{2\mu(1+\nu)\alpha}{1-2\nu} & \text{for 3D or plane strain} \\[2ex] \dfrac{2\mu(1+\nu)\alpha}{1-\nu} & \text{for plane stress} \end{cases} \tag{5}$$

in which ν is the Poisson ratio, μ is the shear modulus and α is the thermal expansion coefficient.

3 Treatment of domain integrals by RIM

Domain integrals in eqs. (1) to (3) can be transformed to boundary by the Radial Integration Method [7]. Particularly, the domain integrals in eqs.(1) and (2) are weakly singular integrals and the RIM can be directly applied, while the domain integral in eq. (3) firstly requires a regularization procedure due to its strongly singular character (see Gao [8, 11]).

The RIM can be summarized by the following expressions:

$$\int_\Omega f(\boldsymbol{\xi}, \mathbf{x})\, d\Omega = \int_\Gamma \frac{1}{r^{\beta-1}}\frac{\partial r}{\partial n}F(\boldsymbol{\xi}, \mathbf{x})\, d\Gamma = \int_\Gamma \frac{r_i n_i}{r^\beta}F(\boldsymbol{\xi}, \mathbf{x})\, d\Gamma \tag{6}$$

where

$$F(\boldsymbol{\xi}, \mathbf{x}) = \int_0^{r(\Gamma)} f(\boldsymbol{\xi}, \mathbf{x}) r^{\beta-1} \, dr \tag{7}$$

in which, $\beta = 2$ for two-dimensional domains and $\beta = 3$ for three-dimensional ones.

The unidimensional integration of eq. (7) is performed along the straight line beginning at the source point, $\boldsymbol{\xi}$, and finishing at a boundary point $\mathbf{x}$. To evaluate such integral, it is necessary to rewrite coordinates x_i (which vary along the mentioned radial straight line) in function of r, i.e., from eq. (4),

$$x_i = \xi_i + r_i = \xi_i + r_{,i} \, r \tag{8}$$

where $r_{,i}$ is a constant in eq. (7) – the direction cosines of $\mathbf{r}$ – but not in eq. (6).

Also, the numerical integration of eq. (7) requires the next parametrization:

$$r = \frac{r(\Gamma)}{2}\eta + \frac{r(\Gamma)}{2}, \qquad (-1 \leqslant \eta \leqslant +1) \tag{9}$$

Since the internal heat source term, $b(\mathbf{x})$, is a known function, the above methodology can be straightforward applied to the domain integral in eq. (1) by simply substituting $\theta^*(\boldsymbol{\xi}, \mathbf{x}) b(\mathbf{x})$ into $f(\boldsymbol{\xi}, \mathbf{x})$.

In the following, the domain integrals in eqs. (2) and (3) are addressed.

3.1 Treatment for the displcament equation domain term

From eq. (5) and considering the Kelvin fundamental solution, it is possible to write the kernel of the last integral in eq. (2) as

$$f_i(\boldsymbol{\xi}, \mathbf{x}) = \epsilon^*_{ijk}(\boldsymbol{\xi}, \mathbf{x}) \sigma^o_{jk}(\mathbf{x}) = \frac{\lambda r_{,i} \, \theta(\mathbf{x})}{(\beta - 1) r^{\beta-1}} \tag{10}$$

where

$$\lambda = \begin{cases} -\dfrac{(1+\nu)\alpha}{2\pi(1-\nu)} & \text{for 3D or plane strain} \\[2ex] -\dfrac{(1+\nu)\alpha}{2\pi} & \text{for plane stress} \end{cases} \tag{11}$$

Each Function $f_i(\boldsymbol{\xi}, \mathbf{x})$, can be considered for RIM equations, such that the radial integration of equation (7) takes the following form:

$$F_i(\boldsymbol{\xi}, \mathbf{x}) = \frac{\lambda r_{,i}}{\beta - 1} \int_0^{r(\Gamma)} \theta(\mathbf{x}(r)) \, dr = \frac{\lambda r_{,i}}{\beta - 1} \int_{-1}^{+1} \theta(\mathbf{x}(r(\eta))) \frac{r(\Gamma)}{2} \, d\eta \tag{12}$$

In evaluation of equation (12), the temperature value is required for each radial integration point. Such information is not directly available, once the temperature is known only at some discrete points, namely the boundary and internal collocation points. Thus, strategies to obtain the temperature at any given domain point are presented later in section 4.

3.2 Treatment for the internal stress equation domain term

As already said, the domain integral in eq. (3) has a strongly singular kernel and needs a regularization step before the RIM transformation. Such a regularization is given in [8] and reproduced below:

$$\int_\Omega \epsilon^*_{ijkl}(\boldsymbol{\xi}, \mathbf{x}) \sigma^o_{kl}(\mathbf{x}) d\Omega = \int_\Omega \epsilon^*_{ijkl}(\boldsymbol{\xi}, \mathbf{x}) [\sigma^o_{kl}(\mathbf{x}) - \sigma^o_{kl}(\boldsymbol{\xi})] d\Omega + \sigma^o_{kl}(\boldsymbol{\xi}) \int_\Omega \epsilon^*_{ijkl}(\boldsymbol{\xi}, \mathbf{x}) d\Omega \tag{13}$$

where the first integral in the right hand side is now weakly singular and can be directly treated by the RIM, while the last integral keeps a strongly singular kernel. However, such an integral can also be transformed to boundary using a spherical (or circular) surface of exclusion around $\boldsymbol{\xi}$ with the radius tending to zero. The result is [11]:

$$\int_{\Omega} \epsilon^{*}_{ijkl}(\boldsymbol{\xi}, \mathbf{x}) d\Omega = \int_{\Gamma} r \frac{\partial r}{\partial n} \ln(r) \epsilon^{*}_{ijkl}(\boldsymbol{\xi}, \mathbf{x}) d\Gamma \tag{14}$$

Since $\boldsymbol{\xi}$ is an internal point, no singularity is present in this boundary integration.

Now, from eq. (5) and the Kelvin fundamental solution, the kernel in the first term on the right hand side of eq. (13), can be written as:

$$f_{ij}(\boldsymbol{\xi}, \mathbf{x}) = \epsilon^{*}_{ijkl}(\boldsymbol{\xi}, \mathbf{x})[\sigma^{o}_{kl}(\mathbf{x}) - \sigma^{o}_{kl}(\boldsymbol{\xi})] = \frac{\gamma(\beta r_{,i}\, r_{,j} - \delta_{ij})[\theta(\mathbf{x}) - \theta(\boldsymbol{\xi})]}{(\beta - 1) r^{\beta}} \tag{15}$$

where $\gamma = 2\mu\lambda$, with λ given in eq. (11).

Thus, the correspondent radial integration for this parcel is

$$F_{ij}(\boldsymbol{\xi}, \mathbf{x}) = \frac{\gamma(\beta r_{,i}\, r_{,j} - \delta_{ij})}{\beta - 1} \int_{-1}^{+1} \left[\frac{\theta(\mathbf{x}(r(\eta))) - \theta(\boldsymbol{\xi})}{r(\eta)} \right] \frac{r(\Gamma)}{2}\, d\eta \tag{16}$$

Note again, the necessity of a procedure to evaluate the temperature at any domain point $\mathbf{x}$, which is discussed in the next section.

4 Temperature evaluation on radial integration points

To evaluate the temperature change at any radial integration point, i.e., $\theta(\mathbf{x})$ in eqs. (12) and (16), an interpolation procedure is required since such values have been obtained only at some discrete points, namely the collocation points designated by $\boldsymbol{\xi}$. In this way, the Moving Least Square (MLS) interpolation procedure [9] can be adopted. As a first step in such procedure, a support domain around the integration point $\mathbf{x}$ is defined in order that only the collocation points inside this region are considered for temperature interpolation, as illustrated in Fig. 1. The range of the support domain is designated by d_w.

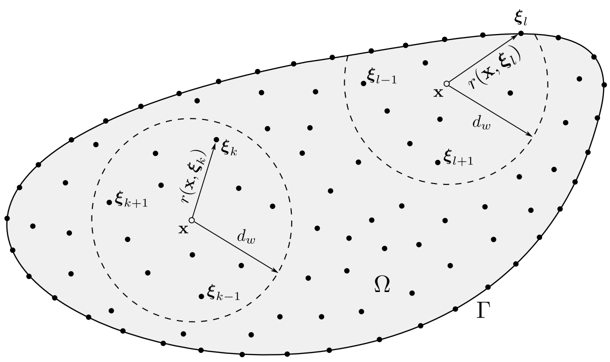

Figure 1. Support domain for temperature evaluation

In the MLS interpolation method, a basis of linearly independent functions is defined. An adequate choice for functions that form such basis are the Pascal triangle monomials. Thus, the projection coefficients of $\theta(\mathbf{x})$ into this basis are obtained from the minimization of a squared residual, weighted by pre-defined functions, considering the known values at some specific points, $\theta(\boldsymbol{\xi}_k)$, where $\boldsymbol{\xi}_k$ $(k = 1, \ldots, n)$ are the collocation points inside the support domain.

Calling the functions that form the referred basis by $\{\phi_j(\mathbf{x})\}$ with $j = 1, \ldots, m$, one can write:

$$\theta(\mathbf{x}) \approx \sum_{j=1}^{m} \alpha_j \phi_j(\mathbf{x}) \tag{17}$$

where α_j are the projection coefficients.

Thus, defining $\theta_k := \theta(\boldsymbol{\xi}_k)$ and $\phi_j^k := \phi_j(\boldsymbol{\xi}_k)$, the weighted-squared residual can be written as

$$\mathcal{R} = \sum_{k=1}^{n} \omega_k \left[\theta_k - \sum_{j=1}^{m} \alpha_j \phi_j^k \right]^2 \tag{18}$$

where $\omega_k := \omega(\mathbf{x}, \boldsymbol{\xi}_k)$ are the weighting functions that have the role to provide favourable weightings for points closer to $\mathbf{x}$.

Some options for the weighting functions are [10]:

Cubic spline (CS) weight function: $$\omega(\mathbf{x}, \boldsymbol{\xi}_k) \equiv \omega(\bar{d}) = \begin{cases} \frac{2}{3} - 4\bar{d}^2 + 4\bar{d}^3 & \text{for } \bar{d} \leqslant \frac{1}{2} \\ \frac{4}{3} - 4\bar{d} + 4\bar{d}^2 - \frac{4}{3}\bar{d}^3 & \text{for } \frac{1}{2} < \bar{d} \leqslant 1 \end{cases} \tag{19a}$$

Quartic spline (QS) weight function: $$\omega(\mathbf{x}, \boldsymbol{\xi}_k) \equiv \omega(\bar{d}) = 1 - 6\bar{d}^2 + 8\bar{d}^3 - 3\bar{d}^4 \tag{19b}$$

Exponential (EXP) weight function: $$\omega(\mathbf{x}, \boldsymbol{\xi}_k) \equiv \omega(\bar{d}) = e^{-(\bar{d}/\kappa)^2} \tag{19c}$$

where κ is a chosen constant and $\bar{d} = \dfrac{r(\mathbf{x}, \boldsymbol{\xi}_k)}{d_w} \equiv \dfrac{|\mathbf{x} - \boldsymbol{\xi}_k|}{d_w}$.

In addition to these cases, a standard (no weighting) MLS procedure can be defined by simple assuming $\omega_k = 1$, for all k.

Minimization of the residual defined in eq. (18) requires that $\dfrac{\partial \mathcal{R}}{\partial \alpha_i} = 0$ (for $i = 1, \ldots, m$), which leads to

$$\sum_{j=1}^{m} \alpha_j \sum_{k=1}^{n} \omega_k \phi_i^k \phi_j^k = \sum_{k=1}^{n} \omega_k \theta_k \phi_i^k \quad \text{for} \quad i = 1, \ldots, m.$$

Rewriting this last result in a matrix form:

$$[A]\{\alpha\} = \{f\} \quad \Rightarrow \quad \begin{cases} [A] \to m \text{ x } m \\ \{\alpha\}, \{f\} \to m \text{ x } 1 \end{cases} \qquad \begin{cases} A_{ij} = \sum_{k=1}^{n} \omega_k \phi_i^k \phi_j^k \\ f_i = \sum_{k=1}^{n} \omega_k \theta_k \phi_i^k \end{cases} \tag{20}$$

Thus, solving eq. (20), the projection coefficients α_j are obtained and can be applied into eq. (17) to evaluate $\theta(\mathbf{x})$. To ensure a non-singular matrix $[A]$, it is important to use a grater number of points in the support domain, compared to the number of basis functions, so that $n > m$. It can be done by implementing an adaptive size algorithm for the support domain.

5 Numerical examples

Two plane state problems with known analytical solutions are used to validate the proposed formulation. The first is a plane strain problem of a thick circular tube with heat generation in its wall and different temperatures inside and outside. The second is a plane stress problem of a beam bending caused by internal heat generation and different temperatures at the upper and lower faces. For both, a Young modulus of $E = 210$ GPa, a thermal conductivity of $k = 52.3$ W/mK and a thermal expansion coefficient of $\alpha = 1.1 \text{ x } 10^{-5}$ K^{-1} were considered. Moreover, for the first example, a Poisson ratio of $\nu = 0.3$ was adopted, while for the second one a null value for this material property was imposed, since the available analytical solution follows the Euler-Bernoulli bending theory.

5.1 Example 1: Long hollow cylinder with heat source

This example is described in Fig. 2 and the adopted mesh is illustrated in Fig. 3 (only a section of 15º was modelled due to symmetry). The analyses have been performed considering $r_1 = 1.2$ m, $r_2 = 2.3$ m, $\theta_1 = 60$ ºC, $\theta_2 = 350$ ºC and $c = 10^6$ W/m^2. The initial range for support domains was set as $d_w = 0.17$ m. The analytical solution was presented by Boley and Weiner [12]. Average percentage errors, considering all (boundary and internal) discrete points, for each weighting function are presented in Table 1. An analysis with standard MLS, i.e., without weighting ($w_k = 1$) was also performed.

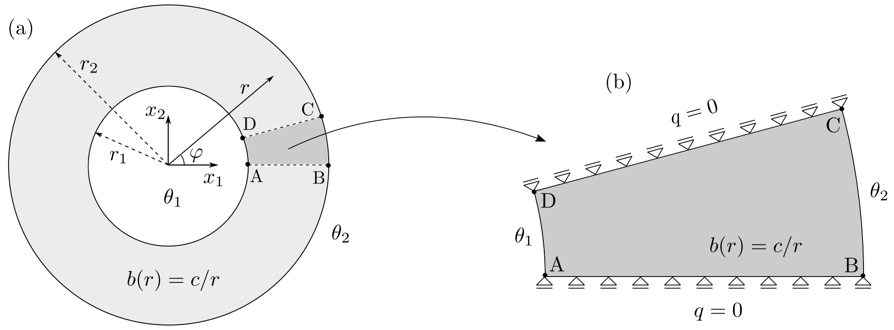

Figure 2. Example 1: Long hollow cylinder; (a) Cross section, (b) Region modelled with boundary conditions

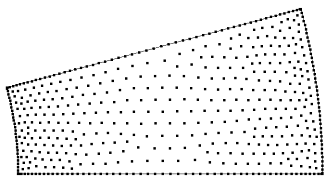

Figure 3. Example 1: Mesh; 87 quadratic boundary elements and 355 internal points

Table 1. Example 1: Average errors for each weighting function

	Std. MLS	CS	QS	EXP
Displacement (u_r) average error	0.0219%	0.0228%	0.0217%	0.0219%
Out-of-plane stress (σ_{33}) average error	0.1173%	0.1326%	0.1240%	0.1261%

5.2 Example 2: Bending of a beam with internal heat source

This example is described in Fig. 4 and the adopted mesh is illustrated in Fig. 5 (only a half was modelled due to symmetry). The analyses have been performed considering $L = 12$ m, $a = 1$ m, $\theta_1 = 70$ ºC, $\theta_2 = -50$ ºC and $c = 6$ x 10^5 W/m^3. The initial range for support domains was set as $d_w = 0.45$ m. The analytical solution was presented by Hetnarski and Eslami [13]. This solution includes the vertical displacements for points located at $x_2 = 0$ (the elastic line) and axial stress distribution along x_2-axis.

For all weighting functions, including the no weighting case, the average error for vertical displacements was 0.005 %. A plot of such results over the analytical line is presented in Fig. 6. Also in Fig. 6, the axial stress results are plotted for points at $x_1 = 3$ m. At points close to $x_1 = 0$, where the single support is placed, the Euler-Bernoulli assumptions are no longer adequate and the stress distribution deviates from the available analytical result. Thus, considering only points located at $x_1 > L/4$, the average error for axial stress was about 0.082 % for all analyses.

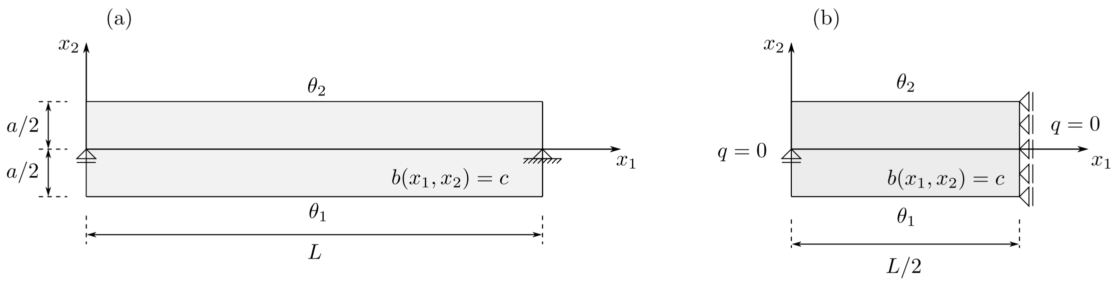

Figure 4. Example 2: Beam with internal heat source; (a) Entire beam, (b) Region modelled

6 Concluding remarks

A cell-less BEM formulation, based on the Radial Integration Method and Meshfree techniques for shape functions evaluation, in particular de Moving Least square procedure, is proposed for uncoupled thermoelastic

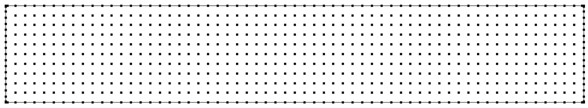

Figure 5. Example 2: Mesh; 76 quadratic boundary elements and 531 internal points

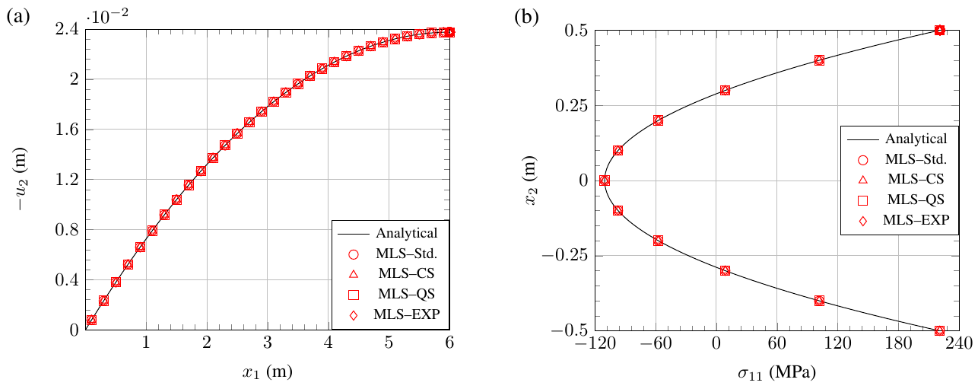

Figure 6. Example 2: (a) Vertical displacements for points at $x_2 = 0$, (b) Axial stress for points at $x_1 = 3$ m

analysis. The Moving Least Square with different weighting functions is used to obtain temperature values at numerical integration points during solution of radial integrals when displacements and stresses are being calculated. The accuracy of the proposed formulation is shown by two plane problems.

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