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A Hybrid–Stabilized FEM Method Applied to Heat Conduction Equation

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Abstract. In this work, it is performed a numerical analysis of a totaly discrete formulation for the transient heat conduction problem. This formulation is constructed by using a discontinuous hybrid stabilized finite element method in space combined with a high order finite difference approximation (Crank-Nicolson method) for the temporal dependency. The computational methodology used to solve the formulation is a static condensation scheme resulting in a global system related only with the Lagrange multiplier associated with the trace of the temperature at the edges of the elements and local problems that are solved for the temperature. In doing so, the number of the degrees of freedom of the global system is reduced. Numerical results are presented confirming the optimal rates of convergence obtained in the numerical analysis.

Keywords: High order methods, Numerical analysis, Heat equation, Parabolic problems, Stabilized methods.

1 Introduction

Transient heat conduction problems are commonly represented by parabolic differential equations and, in the finite element context, the standard approach is the classical Galerkin method usually defined so that the space approximation is continuous between elements, Thomée [1], Fernandes et al. [2]. However, the application of that method together with finite difference time discretization may give rise to spurious oscillations at the initial instants of time, as shown in Harari [3].

Based on the ideas of Arruda et al. [4] for the elliptic problem, a hybrid stabilized finite element formulation for the space discretization combined with a Crank-Nicolson scheme is proposed here in order to eliminate the spurious oscillations, consisting of a generalization of the discontinuous Galerkin (DG) method by adding consistent stabilization terms by introducing a Lagrange multiplier defined at the interfaces of the elements to impose the continuity at the edges.

Hybrid DG methods, Rivière [5], have been developed to improve stability and reduce computational cost comparing with DG methods used alone, Ewing et al. [6], Cockburn et al. [7], maintaining the good properties of the DG methods as, for example, to satisfy constraints locally and allow flexibility for parallel solvers even in mixed formulations. Some advantages of discontinuous interpolations can be seen in Karam-Filho and Loula [8]. The inclusion of unknowns at the interfaces compensates the interelements discontinuity, Arnold et al. [9], allowing the use of static condensation techniques, Lehrenfeld and Schöberl [10], simplifying the solution of the resulting algebraic system.

Then, the method which is analysed in this work is the coupling of local problems where the solution for the primal variable, the temperature, is obtained by a hybrid stabilized DG method, with a global problem that solves for the Lagrange multiplier.

The implementation methodology to solve the resultant formulation uses a static condensation procedure which consists in eliminating the primal variable resulting in a global system relating only the Lagrange multiplier and, once obtained this variable, the temperatures are obtained by solving recovered local systems element by element, reducing in this way the computational cost as in Brezzi et al. [11].

A theoretical numerical analysis of this method is developed in terms of stability and convergence, and numerical results are presented confirming the theoretical results obtained.

2 Model problem

Let $\Omega \subset \mathbb{R}^2$ be an open domain with boundary $\partial\Omega$, u the temperature, $u_0(x,y)$ is the specified initial temperature, $f \in L^2(\Omega)$ a source term, and the time $t \in [0,T]$, $T > 0$. Considering homogeneous Dirichlet boundary conditions, transient heat diffusion problems may be given by the following parabolic linear problem.
Problem 2.1. *Find the temperature $u = u(x,y,t)$, such that*

$$\frac{\partial u}{\partial t} - \mathrm{div}\nabla u = f \qquad \text{in} \qquad \Omega \times [0,T], \tag{1}$$

$$u(x,y,t) = 0 \qquad \text{on} \qquad \partial\Omega \times [0,T]; \quad u(x,y,0) = u_0(x,y) \quad \text{in} \quad \Omega. \tag{2}$$

3 Notations and definitions

Some notations and definitions will be necessary to construct the hybrid stabilized formulation and to develop the numerical analysis. Let $L^2(\Omega) = \left\{ v : \int_\Omega |v|^2 d\Omega < \infty \right\}$, with its usual norm defined by the inner product and represented by $\|\cdot\|_{0,\Omega} = \|\cdot\|_0 = \|\cdot\| = |\cdot|_{0,\Omega} = |\cdot|_0$. We consider spaces of functions mapping the time interval $(0,T)$ to a normed space $L^2(\Omega)$ equipped with the norm $\|\cdot\|$. Thus we define $L^2\left(0,T; L^2(\Omega)\right) = \left\{ z : (0,T) \to L^2(\Omega) : \int_0^T \|z(t)\|^2 dt < \infty \right\}$, Rivière [5]. The finite element partition is given by $\mathcal{T}_h = \{\mathcal{K}\} :=$ $\{$ union of all elements $\mathcal{K}\}$. Moreover, $\mathcal{E}_h$ is the set of all edges e of the elements $\mathcal{K}$, $\mathcal{E}_h^0$ is the set of inner edges and $\mathcal{E}_h^\partial = \mathcal{E}_h \cap \partial\Omega$ is the set of border edges of Ω. Let $[\![\cdot]\!]$ and $\{\cdot\}$ be the jump and the average operators, respectively, defined as in the DG methods, Rivière [5]. Then, given the elements $\mathcal{K}_1, \mathcal{K}_2 \in \mathcal{T}_h$ that share the side e, we define $\mathbf{n}_1, \mathbf{n}_2$ as the unit normal vectors at the edge e of elements $\mathcal{K}_1, \mathcal{K}_2$, respectively, such that for a scalar function φ: $[\![\varphi]\!] = \varphi_1\mathbf{n}_1 + \varphi_2\mathbf{n}_2$ on $e \in \mathcal{E}_h^0$; $[\![\varphi]\!] = \varphi\mathbf{n}$ on $e \in \mathcal{E}_h^\partial$ and $\{\varphi\} = \frac{1}{2}(\varphi_1 + \varphi_2)$ on $e \in \mathcal{E}_h^0$; $\{\varphi\} = \varphi$ on $e \in \mathcal{E}_h^\partial$. Let the broken space, Rivière [5], of finite dimension (in the spatial variable) for the temperature, be given as $V_h^k = \{v_h \in L^2(\Omega) : v_h|_\mathcal{K} \in Q_k(\mathcal{K}), \forall \mathcal{K} \in \mathcal{T}_h\}$, with $Q_k(\mathcal{K})$ the space of polynomial functions of order less or equal to *k* in each variable (quadrilateral elements). For the Lagrange multiplier μ_h that will be introduced in the stabilized formulation, define the space of discontinuous interpolation functions $M_h^l = \{\mu_h \in L^2(\mathcal{E}_h) : \mu_h|_e = p_l(e), \forall e \in \mathcal{E}_h^0,\ \mu_h|_e = 0, \forall e \in \mathcal{E}_h^\partial\}$, where $p_l(e)$ is the space of polynomial functions of order equal or greater than l in each edge e. Let us define the following seminorms which will be necessary to the numerical analysis presented later:

$$|v|_{0,\mathcal{K}}^2 := \int_\mathcal{K} |v|^2 dx;\ |v|_{1,\mathcal{K}}^2 := \int_\mathcal{K} |\nabla v|^2 dx;\ |v|_{2,\mathcal{K}}^2 := \int_\mathcal{K} |\Delta v|^2 dx; \tag{3}$$

$$|v|_{0,h}^2 := \sum_{\mathcal{K}\in\mathcal{T}_h} |v|_{0,\mathcal{K}}^2;\ |v|_{1,h}^2 := \sum_{\mathcal{K}\in\mathcal{T}_h} |v|_{1,\mathcal{K}}^2; \tag{4}$$

$$|v|_*^2 := \sum_{e\in\mathcal{E}_h} h^{-1} \int_e |[\![v]\!]|^2 ds;\ |\mu|_\#^2 := \sum_{e\in\mathcal{E}_h^0} h^{-1} \int_e |\mu|^2 ds; \tag{5}$$

as well as the following norms that have been defined in Arnold et al. [9] and Rivière [5]:

$$\|v\|_{DG}^2 := |v|_{1,h}^2 + |v|_*^2;\ \|\!|v|\!\|_{DG}^2 := \|v\|_{DG}^2 + \sum_{\mathcal{K}\in\mathcal{T}_h} h^2 |v|_{2,k}^2,\ \forall\, [v,\mu] \in V_h^k \times M_h^l. \tag{6}$$

It will be also considered the following norm, previously defined in Arruda et al. [4]:

$$\|[v,\mu]\|_{GC}^2 := \|v\|_{DG}^2 + |\mu - \{v\}|_\#^2, \quad \forall\, [v,\mu] \in V_h^k \times M_h^l\ . \tag{7}$$

Being the finite dimensional product space $V(h) \times M(h)$, where $V(h) = V_h^k + H^2(\Omega) \subset H^2(\mathcal{T}_h)$ and $M(h) = M_h^l + L^2(\mathcal{E}_h^0)$, we will introduce here the alternative norm:

$$|||[v_h, \mu_h]|||^2_{GC} := \|[v_h, \mu_h]\|^2_{GC} + \sum_{\mathcal{K}\in\mathcal{T}_h} h^2 |v_h|^2_{2,\mathcal{K}}, \, \forall [v_h, \mu_h] \in V(h) \times M(h) \tag{8}$$

which can also be rewritten using the definitions (3) as

$$|||[v_h, \mu_h]|||^2_{GC} := |||v_h|||^2_{DG} + |\mu_h - \{v_h\}|^2_{\#}, \quad \forall \, [v_h, \mu_h] \in V(h) \times M(h). \tag{9}$$

Note that (7) and (9) are equivalent since, for $0 < M_1 \leq 1$ and $M_2 < \infty$:

$$M_2 |||[v_h, \mu_h]|||_{GC} \leq \|[v_h, \mu_h]\|_{GC} \leq M_1 |||[v_h, \mu_h]|||_{GC}, \; \forall \, [v_h, \mu_h] \in V_h \times M_h. \tag{10}$$

4 Totally discrete formulation

Based on the ideas found in Arruda et al. [4] for the eliptic problem, a semidiscrete hybrid stabilized formulation for the parabolic problem was proposed and analyzed in Barreiro [12]. Here a new totally discretized method where the hybrid stabilized formulation is used to the space variable combined with a Crank-Nicolson scheme to the time variable will be introduced and analysed for Problem (1–2). This formulation avoid spurious oscilations that arise at small initial times during the simulations when, for example, the continuous Galerkin method is applied in space. Here, a Lagrange multiplier, λ_h, is introduced which is identified with the trace of the primal temperature variable u_h; that is: $\lambda_h = u_h|_e$ in each edge $e \in \mathcal{E}_h$. The boundary condition $u = 0$ on $\partial\Omega$ is weakly imposed by the Nitsche approach usually adopted in the DG methods. A residual term is added making the formulation symmetric and adjoint consistent. It is also added a term for the stabilization of both variables: u_h and the Lagrange multiplier λ_h. The Crank-Nicolson scheme is a kind of *"arithmetic mean"* between the explicit and implicit schemes, giving second-order convergence in time. Here the semidiscrete formulation of Barreiro [12] is discretized in a symmetric fashion around the point $t^{n+\frac{1}{2}} = \left(n + \frac{1}{2}\right) \Delta t = \frac{t^{n+1}+t^n}{2}$.

Let $u_h^n = u_h(t^n)$, $n = 0, \cdots, N-1$, $\Delta t = T/N$ with N being the number of iterations and T the total time, we generate the totally discrete hybrid stabilized formulation for the Problem (1–2):

Problem 4.1. *Find the pair* $\left[u_h^{n+1}, \lambda_h^{n+1}\right] \in V_h^k \times M_h^l$, $\forall [v_h, \mu_h] \in V_h^k \times M_h^l$, $\forall n \geq 0$, *such that*

$$\begin{aligned} \sum_{\mathcal{K}\in\mathcal{T}_h} \frac{1}{\Delta t} \left(u_h^{n+1}, v_h\right)_{\mathcal{K}} + \frac{1}{2} a\left(\left[u_h^{n+1}, \lambda_h^{n+1}\right], [v_h, \mu_h]\right) \\ = \sum_{\mathcal{K}\in\mathcal{T}_h} \frac{1}{\Delta t} \left(u_h^n, v_h\right)_{\mathcal{K}} - \frac{1}{2} a\left(\left[u_h^n, \lambda_h^n\right], [v_h, \mu_h]\right) + F(v_h), \end{aligned} \tag{11}$$

For $m = n+1$ or $m = n$, $a([\cdot,\cdot],[\cdot,\cdot])$ and $F(\cdot)$ can be defined as:

$$\begin{aligned} a([u_h^m, \lambda_h^m], [v_h, \mu_h]) = & \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \nabla u_h^m \cdot \nabla v_h \, dx \; - \int_{\mathcal{E}_h} (\{\nabla v_h\} \cdot [\![u_h^m]\!] + \{\nabla u_h^m\} \cdot [\![v_h]\!]) \, ds \\ & - \int_{\mathcal{E}_h^0} ([\![\nabla v_h]\!](\{u_h^m\} - \lambda_h^m) + [\![\nabla u_h^m]\!](\{v_h\} - \mu_h) \, ds + \int_{\mathcal{E}_h} \frac{\beta_0}{2h} [\![v_h]\!] \cdot [\![u_h^m]\!] \, ds \\ & + \int_{\mathcal{E}_h} \frac{2\beta_0}{h} (\{u_h^m\} - \lambda_h^m)(\{v_h\} - \mu_h) \, ds, \end{aligned} \tag{12}$$

$$F(v_h) = \sum_{\mathcal{K}\in\mathcal{E}_h} \int_{\mathcal{K}} f^{n+\frac{1}{2}} v_h \, dx = \sum_{\mathcal{K}\in\mathcal{E}_h} \int_{\mathcal{K}} \frac{1}{2} \left(f^{n+1} + f^n\right) v_h \, dx. \tag{13}$$

For (12), coercivity and continuity have been obtained by Arruda et al. [4] to the elliptic problem. With the above considerations, we will obtain, in what follows, the stability conditions and *a priori* error estimates.

Lemma 4.1. ***(Stability).*** *For Problem (1–2) there is a constant $\mathcal{C}$ independent of the mesh parameter h and of Δt, such that, $\forall m > 0$,*

$$\|u_h^m\|_{0,h}^2 + \Delta t \mathcal{C} \sum_{n=1}^{m} \| [u_h^n, \lambda_h^n] \|_{GC}^2 \\ \leq \Delta t \mathcal{C} \left[\frac{C_b^2}{4} \sum_{n=1}^{m} \| \left[u_h^{n-1}, \lambda_h^{n-1}\right] \|_{GC}^2 + \frac{M_1^2}{2} \sum_{n=1}^{m} \|f^n + f^{n+1}\|_{0,h}^2 \right] + \mathcal{C} M_1^2 \|u_0\|_{0,h}^2. \tag{14}$$

Proof. Taking $v_h = u_h^{n+1}$, $\mu_h = \lambda_h^{n+1}$ and multiplying by Δt the equation (11); using the Cauchy-Schwarz inequality, the coercivity and continuity of $a([\cdot,\cdot],[\cdot,\cdot])$, using the equivalence of the norms (10), applying Young's inequality, adding all the elements from $n = 0$ to $n = m-1$ and knowing that $u_h^0 = u_0$, we have (14), $\forall\ m \geq 1$, with $\mathcal{C} = \frac{C_s M_2}{4}$, $C_s = \min\left\{ \left(1 - \frac{5C^2}{\beta_0}\right),\ \frac{\beta_0}{4} - \frac{5C^2}{4},\ \beta_0 \right\}$, where C is the constant of the trace inequality, with $C_b = max\ \{1,\ C,\ 2\beta_0\}$, $\gamma_b = C_p\ \|f\|$ and C_p is the Poincaré inequality constant. ☐

Theorem 4.1. ***(Error Estimates).*** *Considering that the exact solution $u(t)$ of Problem (1–2) satisfies $u(t) \in H^1(\mathcal{T}_h)$, $\frac{\partial^2 u(t)}{\partial t^2} \in L^2(\Omega) \quad \forall t \in [0,T]$, there are constants $C_1 = \left(\frac{1}{24C_s}\right)^{1/2}$, $C_2 = \left(\frac{6}{C_s}\right)^{1/2}$, $C_3 = \left(\frac{3C_b}{C_s}\right)^{1/2}$, $C_4 = \left(\frac{1}{12\tilde{C}_s C_s}\right)^{1/2}$, $C_5 = \left(\frac{12}{\tilde{C}_s C_s}\right)^{1/2}$, $C_6 = \left(\frac{C_s}{\tilde{C}_s}\right)^{1/2}$, $C_7 = \left(\frac{6C_b}{C_s\tilde{C}_s}\right)^{1/2}$ and $\tilde{C}_s = M_2 C_s$, with C_s the coercivity constant and C_b the continuity constant, independents of h and Δt, $\forall m > 0$, such that*

$$\|e_u^m\|_{0,h} \leq \Delta t^2 C_4 \left(\int_0^{t^m} \left\| \frac{\partial^3 u^n}{\partial t^3} \right\|_{0,h}^2 dt \right)^{\frac{1}{2}} + C_5 h^{k+1} \left(\frac{1}{\Delta t} \int_0^{t^m} \left| \frac{\partial u^n}{\partial t} \right|_{k+1,\Omega}^2 dt \right)^{\frac{1}{2}} \\ - C_6 \left(\sum_{n=1}^{m-1} \| [e_u^n, e_\lambda^n] \|_{GC}^2 \right)^{\frac{1}{2}} + C_7 \left(\sum_{n=2}^{m} \| \left[e_u^{n-1}, e_\lambda^{n-1}\right] \|_{GC}^2 \right)^{\frac{1}{2}}; \tag{15}$$

$$\left(C_s \sum_{n=1}^{m} \left\| \left[u^{n+1} - u_h^{n+1}, \lambda^{n+1} - \lambda_h^{n+1}\right] \right\|_{GC}^2 \right)^{1/2} \leq C_1 \Delta t^2 \left(\int_0^{t^m} \left\| \frac{\partial^3 u^n}{\partial t^3} \right\|_{0,h}^2 dt \right)^{1/2} \\ + C_2 h^{k+1} \left(\frac{1}{\Delta t} \int_0^{t^m} \left| \frac{\partial u^n}{\partial t} \right|_{k+1,\Omega}^2 dt \right)^{1/2} + C_3 \left(\sum_{n=1}^{m} \| \left[e_u^{n-1}, e_\lambda^{n-1}\right] \|_{GC}^2 \right)^{1/2}, \tag{16}$$

with $e_u = u_h(t) - \tilde{u}_h(t)$ and $e_\lambda = \lambda_h(t) - \tilde{\lambda}_h(t)$ where $\tilde{u}$ and $\tilde{\lambda}$ the elliptic projections of u and λ, respectively, which define $\tilde{u}_h(t)$ and $\tilde{\lambda}_h(t)$ as $u(t) - u_h(t) = u(t) - \tilde{u}_h(t) + \tilde{u}_h(t) - u_h(t) = \rho_u(t) - e_u(t)$ and $\lambda(t) - \lambda_h(t) = \lambda(t) - \tilde{\lambda}_h(t) + \tilde{\lambda}_h(t) - \lambda_h(t) = \rho_\lambda(t) - e_\lambda(t)$. It is used $u^n = u(t^n)$, $\lambda^n = \lambda(t^n)$, $\tilde{u}^n = \tilde{u}(t^n)$ and $\tilde{\lambda}^n = \tilde{\lambda}(t^n)$.

Proof. From the consistecy of (4.1) and using the elliptic projection defined, we arrive at

$$\sum_{\mathcal{K} \in \mathcal{T}_h} \left(\frac{e_u^{n+1} - e_u^n}{\Delta t}, v_h \right)_{\mathcal{K}} + \frac{1}{2} a\ \left(\left[e_u^{n+1}, e_\lambda^{n+1}\right], [v_h, \mu_h] \right) + \frac{1}{2} a\ \left([e_u^n, e_\lambda^n], [v_h, \mu_h] \right) \\ = \sum_{\mathcal{K} \in \mathcal{T}_h} \left(\frac{\partial u^{n+\frac{1}{2}}}{\partial t} - \frac{u^{n+1} - u^n}{\Delta t}, v_h \right)_{\mathcal{K}} + \sum_{\mathcal{K} \in \mathcal{T}_h} \left(\frac{\rho_u^{n+1} - \rho_u^n}{\Delta t}, v_h \right)_{\mathcal{K}}. \tag{17}$$

Taking $v_h = e_u^{n+1}$ and $\mu_h = e_\lambda^{n+1}$ and by using the coercivity and the continuity of $a\left([\cdot,\cdot],[\cdot,\cdot]\right)$, defining $\theta^{n+\frac{1}{2}} = \dfrac{\partial u^{n+1}}{\partial t} - \dfrac{(u^{n+1} - u^n)}{\Delta t}$, using Cauchy-Schwarz inequality, knowing that $\left\|e_u^{n+1}\right\|_{0,h} \leq \left\|\left[e_u^{n+1}, e_\lambda^{n+1}\right]\right\|_{GC}$ and applying the equivalence of the norms (10) we obtain

$$\frac{1}{2\Delta t}\left(\|e_u^{n+1}\|_{0,h}^2-\|e_u^n\|_{0,h}^2\right)+\frac{C_s}{4}\vert\!\vert\!\vert \left[e_u^{n+1},e_\lambda^{n+1}\right]\vert\!\vert\!\vert_{GC}^2 \leq \frac{3}{C_s}\left(\|\theta^{n+\frac{1}{2}}\|_{0,h}^2+\left\|\frac{\rho_u^{n+1}-\rho_u^n}{\Delta t}\right\|_{0,h}^2\right)+\frac{3C_b}{2C_s}\vert\!\vert\!\vert \left[e_u^n,e_\lambda^n\right]\vert\!\vert\!\vert_{GC}^2. \tag{18}$$

Performing Taylor expansion for $\theta^{n+\frac{1}{2}}$ and for ρ_u^{n+1}, using Cauchy-Schwarz inequality, adding all the terms from $n=0$ until $n=m-1$, knowing that $\|e_u^0\|=0$, since $u_h^0=\tilde{u}_h^0=u_0$ and using the result for the error estimate $\|u-u_h\|_{0,\Omega}\leq Ch^{k+1}|u|_{k+1,\Omega}$, we obtain

$$\frac{1}{\Delta t}\|e_u^m\|_{0,h}^2+\frac{C_s}{2}\sum_{n=1}^{m}\vert\!\vert\!\vert \left[e_u^n,e_\lambda^n\right]\vert\!\vert\!\vert_{GC}^2 \leq \frac{\Delta t^4}{24C_s}\int_0^{t^m}\left\|\frac{\partial^3 u^n}{\partial t^3}\right\|_{0,h}^2 dt + \frac{6}{\Delta t C_s}h^{2(k+1)}\int_0^{t^m}\left|\frac{\partial u^n}{\partial t}\right|_{k+1,\Omega}^2 dt+\frac{3C_b}{C_s}\sum_{n=1}^{m}\vert\!\vert\!\vert \left[e_u^{n-1},e_\lambda^{n-1}\right]\vert\!\vert\!\vert_{GC}^2. \tag{19}$$

From the above results, in what follows, the estimates of Theorem 4.1 will be obtained.

(i) Obtaining Estimate (15) ($L^2(\Omega)$-norm): From (19), the norms equivalence, being $\vert\!\vert\!\vert \left[e_u^0,e_\lambda^0\right]\vert\!\vert\!\vert_{GC}=0$, since $u_h^0=\tilde{u}_h^0=u_0$ and $\lambda_h^0=\tilde{\lambda}_h^0=\lambda_0$, we obtain

$$\frac{\tilde{C}_s}{2}\|e_u^m\|_{0,h}^2 \leq \frac{\Delta t^4}{24C_s}\int_0^{t^m}\left\|\frac{\partial^3 u^n}{\partial t^3}\right\|_{0,h}^2 dt + \frac{6}{\Delta t C_s}h^{2(k+1)}\int_0^{t^m}\left|\frac{\partial u^n}{\partial t}\right|_{k+1,\Omega}^2 dt - \frac{C_s}{2}\sum_{n=1}^{m-1}\vert\!\vert\!\vert \left[e_u^n,e_\lambda^n\right]\vert\!\vert\!\vert_{GC}^2+\frac{3C_b}{C_s}\sum_{n=2}^{m}\vert\!\vert\!\vert \left[e_u^{n-1},e_\lambda^{n-1}\right]\vert\!\vert\!\vert_{GC}^2. \tag{20}$$

Taking the square root, we obtain (15).

(ii) Obtaining Estimate (16) (Energy norm): From the definition of $\|\cdot\|_{GC}$, since $\left\|\left[\rho_u^{n+1},\rho_\lambda^{n+1}\right]\right\|_{GC}^2=0$, adding from $n=0$ to $n=m-1$, using the equivalence of norms (10) and substituting (19), we get

$$C_s\sum_{n=1}^{m}\left\|\left[u^{n+1}-u_h^{n+1},\lambda^{n+1}-\lambda_h^{n+1}\right]\right\|_{GC}^2 \leq C_s\sum_{n=1}^{m}\vert\!\vert\!\vert \left[e_u^{n+1},e_\lambda^{n+1}\right]\vert\!\vert\!\vert_{GC}^2 \leq \frac{C_s}{2}\sum_{n=1}^{m}\vert\!\vert\!\vert \left[e_u^n,e_\lambda^n\right]\vert\!\vert\!\vert_{GC}^2 \leq \frac{\Delta t^4}{24C_s}\int_0^{t^m}\left\|\frac{\partial^3 u^n}{\partial t^3}\right\|_{0,h}^2 dt + \frac{6}{\Delta t C_s}h^{2(k+1)}\int_0^{t^m}\left|\frac{\partial u^n}{\partial t}\right|_{k+1,\Omega}^2 dt+\frac{3C_b}{C_s}\sum_{n=1}^{m}\vert\!\vert\!\vert \left[e_u^{n-1},e_\lambda^{n-1}\right]\vert\!\vert\!\vert_{GC}^2. \tag{21}$$

Taking the quare root of (21), we obtain (16).

□

5 Hybrid Solver

Since $v_h\in V_h^k$ is independently defined in each element $\mathcal{K}\in\mathcal{T}_h$ and considering discontinuous interpolation, we can eliminate the degrees of freedom relative to the primal variable and the totaly discrete formulation can be solved by using a *Static Condensation* technique which consists, in this case, of solving the problem in two steps: one that solves a global system defined in $\mathcal{E}_h$, obtaining the Lagrange multipliers, by eliminating the temperature variable, and the other one by solving for the temperature in each element $\mathcal{K}$ through local problems, once known the Lagrange multipliers by the previous step. This is done as below where the local and the global systems have been written in a form without the jumps and averages. Then, for all $n=1,...,N$ with $\Delta t=T/N$:

Local Problems: Find $u_h(t)\in V_h^k(\mathcal{K})=V_h^k\big|_{\mathcal{K}}$, such that, $\forall\ v_h\big|_{\mathcal{K}}\in V_h^k(\mathcal{K})$,

$$\int_{\mathcal{K}} \left(\frac{u_h^{n+1} - u_h^n}{\Delta t} \right) v_h \; dx \; + \int_{\mathcal{K}} \frac{1}{2} \left(\nabla u_h^{n+1} + \nabla u_h^n \right) \cdot \nabla v_h \; dx - \int_{\partial\mathcal{K}} \frac{1}{2} \left(\nabla u_h^{n+1} + \nabla u_h^n \right) \cdot \mathbf{n}_{\mathcal{K}} \; v_h \; ds$$
$$+ \; \epsilon \int_{\partial\mathcal{K}} \nabla v_h \cdot \mathbf{n}_{\mathcal{K}} \left(\frac{u_h^{n+1} + u_h^n}{2} - \frac{\lambda_h^{n+1} + \lambda_h^n}{2} \right) \; ds$$
$$+ \int_{\partial\mathcal{K}} \frac{\beta_0}{h} \left(\frac{u_h^{n+1} + u_h^n}{2} - \frac{\lambda_h^{n+1} + \lambda_h^n}{2} \right) v_h \; ds \; = \; \int_{\mathcal{K}} \left(\frac{f^{n+1} + f^n}{2} \right) \; v_h \; dx. \tag{22}$$

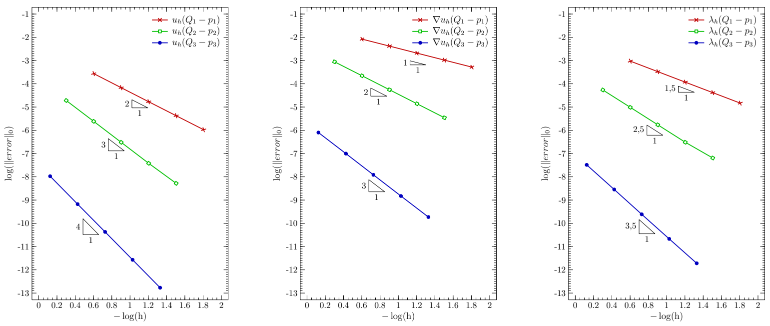
Global Problem: Find $\lambda_h(t) \in M_h^l$, such that, $\forall \mu_h \in M_h^l$,

$$\sum_{\mathcal{K} \in \mathcal{T}_h} \Big[\int_{\partial\mathcal{K}} \frac{1}{2} \left(\nabla u_h^{n+1} + \nabla u_h^n \right) \cdot \mathbf{n}_{\mathcal{K}} \; \mu_h \; ds - \int_{\partial\mathcal{K}} \frac{\beta_0}{h} \left(\frac{u_h^{n+1} + u_h^n}{2} - \frac{\lambda_h^{n+1} + \lambda_h^n}{2} \right) \mu_h \; ds \; \Big] \; = \; 0. \tag{23}$$

It can be noticed that one may adopt any order l of interpolation functions (continuous or discontinuous) for the multiplier λ_h^{n+1} independently of the order k adopted to u_h^{n+1}. Here, only discontinuous interpolation functions will be considered for the multiplier.

6 Numerical Results

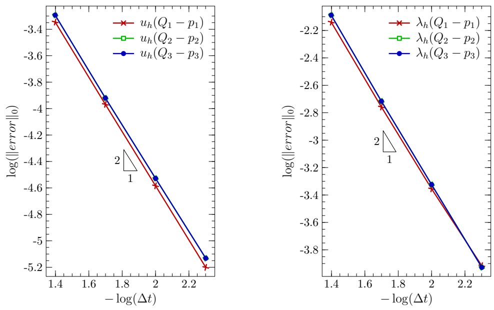
In this section, it will be presented a study of h-convergence as well as of Δt-convergence in the $L^2(\Omega)$-norm for the hybrid stabilized parabolic method presented. The experiments have been performed for uniform quadrilateral meshes defined in a bi-dimensional domain $(2D) = [0,1] \times [0,1]$. It was considered the initial condition $u_0 = 0$ and the source term has been defined as $f(x,y) = sen(\pi x) sen(\pi y)$, such that $u(x,y,t) = \left[\frac{1}{2\pi^2} - \frac{1}{2\pi^2} e^{-2\pi^2 t} \right] sen(\pi x) sen(\pi y)$ is the exact solution for the problem. Same order interpolations have been used for the temperature and the multiplier, considering $Q_1 - p_1$, linear, $Q_2 - p_2$, quadratic and $Q_3 - p_3$, cubic elements. For all the cases it was considered $\epsilon = -1$, giving a symmetric formulation. It has been set $\beta_0 = 10, 16$ and 24 for the cases $Q_1 - p_1$, $Q_2 - p_2$ and $Q_3 - p_3$, respectively.



(a) u_h (b) ∇u_h (c) λ_h

Figure 1. h-convergence for u_h, ∇u_h and λ_h in $L^2(\Omega)$-norm, for elements $Q_1 - p_1$, $Q_2 - p_2$, $Q_3 - p_3$, $\Delta t = 10^{-4}$.

Figure 1 presents the results of the h-convergece study performed, for u_h^n, ∇u_h^n and λ_h^n, in the $L^2(\Omega)$-norm considering $\Delta t = 10^{-4}$ and $T = 0.2$. It can be observed that in all the cases optimal convergence rates have been obtained, that is: order $\mathcal{O}(h^{k+1})$ for u_h, order $\mathcal{O}(h^k)$ for ∇u_h and order $\mathcal{O}(h^{k+0.5})$ for the multiplier (λ_h). A Δt-convergence study can be seen in Figure 2, to which it has been fixed a 64×64 mesh. Optimal convergence orders of $\mathcal{O}(\Delta t^2)$ have been obtained.



(a) u_h (b) λ_h

Figure 2. Δt-convergence for u_h, λ_h in $L^2(\Omega)$-norm, for elements $Q_1 - p_1$, $Q_2 - p_2$, $Q_3 - p_3$.

7 Conclusions

In this work, it has been done a numerical analysis for a hybrid stabilized finite element method, applied to transient heat conduction problems combined with a Crank-Nicolson scheme dealing with the time dependency. Stability and convergence estimates have been obtained independently of the mesh parameter h. Since discontinuous interpolations used, it was possible to use a computational technique based on static condensation, reducing the computational cost by the reduction in the number of degrees of freedom. The numerical experiments confirmed the orders of convergence obtained by the analysis and it was evidenced the role of the stabilization β_0 parameter, since the solutions have been obtained here free of spurious oscillations even for very small Δt's.

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