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Dynamic analysis of prestressed concrete beams in service

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Abstract. The study and improvement of research related to the structural dynamics are more relevant to each development leap of new techniques and materials used in constructing new buildings. These buildings have been designed with a bolder look due to advanced technical regulations for using concrete and steel with high strength. These constructions have become more susceptible to actions that can generate significant vibrations in the structure and make them increasingly efficient, which requires intelligent vibration control. Allied to these factors, the evolution of theoretical models allows an accurate analysis of the system, which architects and structural engineers widely explore. The dynamic analysis of concrete beams submitted to these actions is justified because these elements undergo a high degree of bending during service, resulting in a significant stiffness loss by cracking. The prestressing applied to these structures has become an effective and widely used technology, avoiding, in most cases, the development of tensile stresses, allowing the maintenance of stiffness necessary for vibration control during service. This work investigates the dynamic structural behavior of a reinforced concrete beam during its service. Its behavior is studied due to vertical loads in operation and axial loads from the prestressing action. Besides, the influence of the fundamental dynamic parameters in beam behavior is verified due to these actions. The results obtained are compared with the recommendations prescribed in the national and international codes.

Keywords: NBR 6118, natural frequencies, analytical solution, numerical solution.

1 Introduction

The analysis and improvement of research allied to the tools related to structural dynamics are more relevant with the advent of new construction techniques and materials used in buildings. Vertical towers are increasingly slender, becoming more susceptible to lethal effects related to vibrations. In addition to these factors, the evolution of theoretical models allows an accurate structure analysis, a fact widely explored by designers during special structure projects.

In this context, problems may arise related to excessive vibration of structures when subjected to periodic excitations in service [1]. This structural non-conformity may violate the limits established in the Service Limit State (SLS) and may even reach the Ultimate Limit State (ULS) by excessive local or global vibrations of buildings, cases provided for in NBR 6118 [2].

This state of excessive vibrations in the buildings is amplified in reinforced concrete structures because of the phenomenon of concrete cracking, especially in parts passively armed and subjected to bending efforts. Cracking of these elements is inevitable, even under service actions [2]. According to Abeele and Visscher [1] and Maas et al. [3], the fracture of the reinforced concrete structures occurs for loadings well below service loads, accentuating the nonlinear behavior and reducing the values of their natural frequencies.

Ibrahem [4] studied the cracked Euler-Bernoulli beam concerning buckling and dynamic stability. This author found that a considerable redistribution of stresses occurred in addition to the stiffness loss and vibration effects accentuated. Besides, the structure's natural frequencies undergo significant amplitude variations due to the cracking process. Christides and Barr [5] verified a cracked Euler-Bernoulli beam supported at the ends subjected to vibration. These authors compared experimental results with a theoretical formulation in which the local perturbation in the stresses due to cracks is inserted as a function with an exponential decay law. The results match.

Currently, nondestructive testing has been widely used worldwide to detect cracks and the stiffness loss of structures. These tests consist of the determination of the natural frequencies of the system and their respective mode shape. When a structure is damaged, its frequencies change, and the vibration modes are disturbed [6], [7], [8]. These methods are generally low-cost and practical to be employed.

Thus, when in SLS, slender or not cracked buildings should be dynamically checked, considering their realistic stiffness. That is, considering the physical nonlinearity of the sections. The case is closely linked to excessive deformations and users' comfort. These ideas confront the NBR 6118 [2] recommendations, which allow vibration analysis in a linear regime. In this case, the study of reinforced concrete buildings may present significant distortions with reality due to concrete cracking.

The same document presents alternatives to mitigate the cracking of concrete structures, citing the use of steel previously tensioned the prestressed. According to Schmid [9], among the pertinent advantages of using this mechanism in concrete structures is the elimination/reduction of cracking and combating material fatigue because there is an attenuation of the process of stress inversion. These benefits result from the fact that the section is continuous and subjected to compression action, being possible to consider the integrality of the section stiffness during the analysis, either static or dynamic, in the vast majority of cases.

Many authors [10], [11], [12] investigate beams' natural frequencies and mode shapes submitted to prestressed. These authors concluded that axial loadings alter the natural frequencies to some degree. According to Saiidi [12], axial loadings can potentially affect only the lowest natural frequencies.

The main objective of this paper is to theoretically investigate the dynamic properties of the Euler-Bernoulli beam submitted to an axial compression loading generated by prestressed steel. It is organized into six topics. Topic 2 describes the physical model of the problem and the mathematical formulation of the analytical solution that describes its mechanical behavior. Topic 3 presents the normative recommendations for the practical solution to the problem. Topic 4 discusses the methodology and its validation. In topic 5, the main results are presented and discussed. Finally, topic 6 sets out the main conclusions.

2 Theoretical Basis

Figure 1 shows a Euler-Bernoulli beam, perfectly straight, inextensible of length L, with a rectangular section and hinged ends (A and B). This beam is subject to an axial compressive load P. Consider that beam has the uniformly distributed mass (ρ) per unit length, linear elasticity modulus E, and inertia moment I. Vertical deflection is described by parameter $v(x,t)$, which equilibrium movement equation is expressed as:

$$\rho\frac{\partial^2 v(x,t)}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 v(x,t)}{\partial x^2}\right)+P\frac{\partial^2 v(x,t)}{\partial x^2}=0. \tag{1}$$

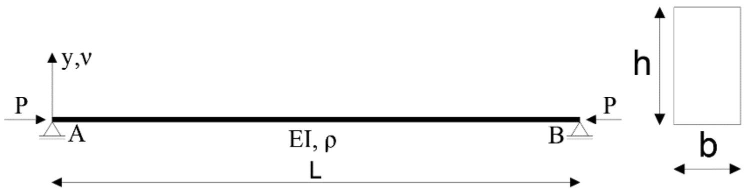

Figure 1. Euler-Bernoulli beam

Consider that the beam has the bending stiffness EI constant, Eq. (1) is rewritten as

$$\rho\frac{\partial^2 v(x,t)}{\partial t^2}+EI\frac{\partial^4 v(x,t)}{\partial x^4}+P\frac{\partial^2 v(x,t)}{\partial x^2}=0. \tag{2}$$

2.1 Solution of movement equation

Equation (2) is the movement equation of the beam in free vibration, whose solution can be found in the literature [13]. In addition, natural frequencies and mode shapes can be obtained. So, the circular natural frequencies (rad/s) are given by:

$$\omega_n^2 = \frac{n^4\pi^4}{L^4}\frac{EI}{\rho}\left(1 - \frac{P}{n^2 P_{cr}}\right) \rightarrow n = 1,2,3,\cdots \tag{3}$$

P_{cr} is the lowest Euler buckling load for a simply supported beam.

$$P_{cr} = \frac{\pi^2 EI}{L^2} \tag{4}$$

When P is zero in Eq. (3), the frequency is the same as that of a simply supported beam with uniformly distributed mass. This frequency is written as

$$\omega_0 = \frac{n^2\pi^2}{L^2}\sqrt{\frac{EI}{\rho}} \rightarrow n = 1,2,3,\cdots \tag{5}$$

Taking the square root of Eq. (3) and dividing by Eq. (5), the following equation is obtained:

$$\frac{\omega_n}{\omega_0} = \sqrt{1 - \frac{P}{n^2 P_{cr}}} \rightarrow n = 1,2,3,\cdots \tag{6}$$

This expression allows the evaluation of the beam's natural frequency reduced by the axial load. Figure 2 shows the curves described by Eq. (6) for modes n = 1, 2, 3, and 8. It is observed that for all vibration shapes, $\omega_n = \omega_0$ when P = 0; the natural frequency of the beam is reduced as P increases; when P equals the critical load of the respective mode n, $\omega_n = 0$. In the last case, the phenomenon can be interpreted as a vibration problem with an infinite period. I.e., the beam is in equilibrium in a slightly deflected way [14]. Moreover, it can be said that for lower modes, the frequency reduction is faster than for the higher shapes.

It is still verified by Eq. (6) that if P is lower than P_{cr}, the natural frequencies will always be positive. Otherwise, the frequencies will be imaginary, limiting the system to small oscillations.

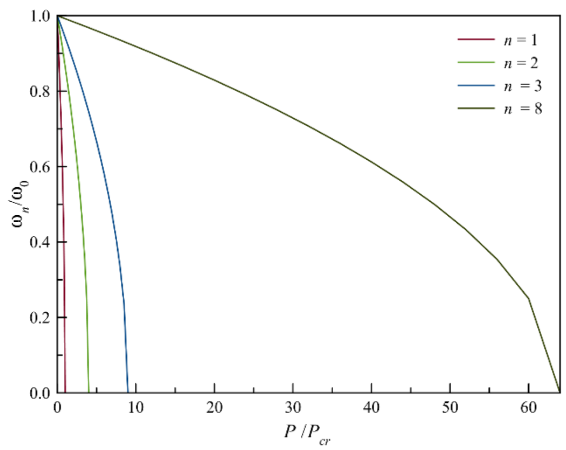

Figure 2. Relationship between natural frequencies and axial loading

3 Design Normative Criteria

Reinforced concrete elements have low tensile strength, making cracks appear in this element. These cracks, if excessive, can compromise the concrete durability, which should be controlled in design. This control is carried out in the service limit state. So, the crack formation can be verified when the maximum tensile stress of the concrete in state I (linear elastic behavior) reaches the tensile strength in bending (f_{ct}), which occurs when the service moment achieves the cracking moment (M_r). NBR 6118 [2] defines this moment as:

$$M_r = \frac{\alpha f_{ct} I_c}{y_t} \tag{7}$$

α is the factor that correlates tensile strength in bending with direct tensile strength. Its value follows the section typology, being 1.5 for a rectangular section. f_{ct} is the direct tensile strength of the concrete. For crack formation cases, this parameter is defined as: $f_{ct} = 0.21 f_{ck}^{2/3}$. f_{ck} is the characteristic compressive strength of the concrete, I_c is the inertia moment of the gross concrete section, and y_t is the distance from the gravity center to the most tensioned fiber.

When the soliciting moment of the beam is greater than the cracking moment, its initial stiffness (EI) is reduced with the crack opening and when the material is in a plastic regime. Then the stiffness needs to be corrected, whose expression recommended by NBR 6118 [2] is given by:

$$EI_{eq} = E_{cs} \left\{ \left(\frac{M_r}{M_{\max}} \right)^3 I_c + \left[1 - \left(\frac{M_r}{M_{\max}} \right)^3 \right] I_{II} \right\} \le E_{cs} I_c \tag{8}$$

where (EI)$_{eq}$ is the equivalent stiffness of the cracked beam; $M_{\max}$ is the maximum bending moment; I_{II} is the inertial moment of the section in state II (fissured); E_{cs} is the secant elasticity modulus of concrete that is given by

$$E_{cs} = \alpha_i E \Rightarrow E = \alpha_E 5600 \sqrt{f_{ck}} \text{ (MPa)} \tag{9}$$

α_E depends on the aggregate type used for concrete, equal to the unit when using granite and gneisses. α_i depends on the concrete compression strength, being 0.85 if f_{ck} = 20 MPa and 0.88, if f_{ck} = 30 MPa.

The ABECE/IBRACON technical committee [15] recommends an adaptation of Eq. (9) to determine the equivalent stiffness of beams. This adaptation consists of a weighted average between the regions with and without fissures. For a simply supported beam, the bending moment close to the supports did not yet exceed the cracking moment, and the equivalent stiffness is calculated more accurately. Thus, the weighted equivalent stiffness $(EI)_{eq,w}$ is expressed as

$$EI_{eq,w} = \frac{1}{L} \quad EI_{eq,1} a_1 + EI_{eq,2} a_2 + EI_{eq,3} a_3 \quad \le E_{cs} I_c \tag{10}$$

where $(EI)_{eq,1}$ and $(EI)_{eq,3}$ are the equivalent stiffness of the non-cracked section and their respective lengths, a_1 and a_3. $(EI)_{eq,2}$ is the equivalent stiffness of the beam cracked section, with its length a_2. This equivalent stiffness should be calculated in each region using Eq. (8), and M_{max} may be substituted by the respective bending moment [16]. This process is used to evaluate the beam's natural frequencies during its service state without axial loading, i.e., in its cracked form.

NBR 6118 [2] limits the maximum compression force to be applied in concrete structures at any lifetime to 70% of the design f_{ck} and imposes a minimum compression of 1.0 MPa for prestressed structures. This restriction criterion defines the maximum value in the compression module applied on the beam. Furthermore, the recommendation that beam sections maintain good ductile behavior by limiting the position of the neutral line is respected.

4 Methodology

The analysis methodology evaluates an isostatic beam in equilibrium subjected to vertical and axial loads. Firstly, the beam is analyzed theoretically according to the prescriptions described in item 3. The dynamic numerical analysis is performed in the software ADAPT-Builder 2019. This software is based on the finite element method to analyze reinforced and prestressed concrete structures. This software considers the effect of cracking in reinforced concrete sections on the finite element stiffness matrix and axial loadings in its dynamic analysis. The fundamental dynamic parameters of the beam, with and without horizontal efforts, are evaluated and compared with the results found and related normative recommendations used in building projects.

4.1 Validation

The experimental results obtained by Veríssimo [17] are used to validate the proposed methodology. This author realized a dynamic test of a beam in reinforced concrete with the same boundary conditions shown in Fig. 1, identifying its first natural frequencies.

The prismatic beam of 3.0 m in length has a transversal section constant of 14 x 30 cm with a specific weight of 25 kN/m³, of which the concrete is C25. The longitudinal elasticity modulus is 33.03 GPa. The diameter of longitudinal steel rebars is 12.5 mm, being 2 bars. The gravity center is located at 4.1cm from the section's superior face, resulting in an inertia moment of 6562.92 cm^4 for the homogenization of the fissured region. Notably, this value of I_{II} is about 20% of I_c. Analytical results are compared with Veríssimo's results (Tab. 1).

Table 1. Comparison of natural frequencies

Mode Shapes	Natural Frequencies (Hz)					
	Uncracked Analytical	Cracked [17]	Relative error (%)	Uncracked Analytical	Cracked [17]	Relative error (%)
1	56.98	57.30	0.56	42.09	49.51	17.65
2	227.91	229.18		168.35	198.06	
3	512.80	515.66		378.79	445.63	
4	911.64	916.74		673.41	792.24	

For the uncracked section case, the results are satisfactory, with an error approximate of 0.56% for all modes, since the geometric and physical parameters remain the same. According to Timoshenko [14], this difference is due to the deformation contributions generated by the shearing.

For the cracked section, the error observed is not acceptable, but this result is due to the significant difference between the inertias considered (I_{II} = 20%I_c). Carvalho [18] estimates that the equivalent inertia is approximately 50% of the gross inertia for concrete beams in buildings during the service.

5 Case Study

The case study considers a concrete beam (Fig. 1) idealized with characteristic compressive resistance of 30 MPa, specific weight of 2500 kg/m³, length L of 5.0 m, and b = 20 cm and h = 50 cm. The concrete elasticity modulus is determined by Eq. (9).

Longitudinal steel rebars (Fig. 3) are composed of 3 bars CA50 with a diameter of 16 mm, and a gravity center located at 3.5 cm from the section's superior face. The steel elasticity modulus is 210 GPa. Besides, the stirrup mechanical contributions and any other steel parts on the beam are disregarded.

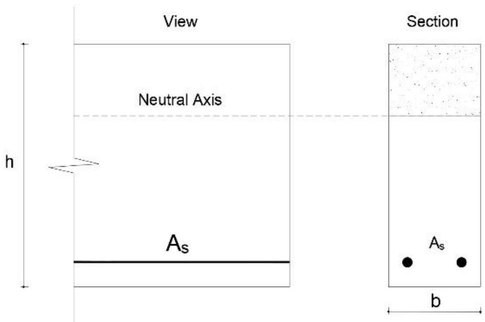

Figure 3. Arrangement of geometric beam data

The permanent load on the beam is balanced with 4 CP190 RB steel cables (elasticity modulus of 200 GPa). The anchorage system is located in the section gravity center, with parabolic tendons whose lower point is placed 8 cm from the beam bottom.

5.1 Results and discussion

The cracking moment (Eq. 7) is approximately 2.53 tf·m (24.81 kN·m), and the maximum moment of the beam in service is 8.03 tf.m (78.75 kN·m), which makes it necessary to find an equivalent stiffness for any minimally adequate analysis. Equations (8) and (10) verify that the stiffness in service is approximately 44.3% of the uncracked stiffness.

The objective of prestressed (axial loading) is to compress the section permanently, allowing maintain the intact rigidity of the beam. Table 2 shows the analytical results of the first natural frequencies with and without consideration of the effect of axial loading P.

Table 2. Analytical results of natural frequencies of the beam

Mode Shapes	**Natural Frequencies (Hz)**				
	Uncracked	**Cracked**	**Relative error (%)**	**Uncracked**	**1 – P/(n^2P$_{cr}$)**
	P = 0			**P ≠ 0**	
1	29.43	19.58	33.47	23.03	0.78
2	117.70	78.32		92.11	
3	264.83	176.21		207.24	
4	470.81	313.27		368.43	

The analytical results presented in the Tab. 2 show that after the cracking process, there is a considerable reduction in the natural frequency values. This reduction is approximately 33%, indicating the structure's flexibilization by the section's cracking. Already the presence of the pretension (P ≠ 0) reduces the frequency values but maintains the stiffness intact, with a minimum force that prevents the formation of cracks. This behavior was observed by Timoshenko [14] and Chen [19].

The axial force applied in this example is 48 tf, generating stress of approximately 5 MPa in the section. This value is lower than the normative limit of 21 MPa. This limit value is improbable in the service in practice since most of the force losses in the cables have already occurred. If it were possible to reach a compression of 21 MPa, the critical load of Euler would be exceeded. On the other hand, minimum stress of 1.0 MPa would reduce 5% the natural frequencies values.

The numerical analysis discretizes the beam with beam elements of 0.3 m size, which uses Hermite polynomials with Gauss integration (full integration). Table 3 shows the numerical results of natural frequencies. These values are slightly higher than the theoretical values, which was expected. However, they present the same behavior observed in the analytical results.

Table 3. Numerical results of natural frequencies of the beam

Mode Shapes	**Natural Frequencies (Hz)**				
	Uncracked	**Cracked**	**Relative error (%)**	**Uncracked**	**1 – P/(n^2P$_{cr}$)**
	P = 0			**P ≠ 0**	
1	30.01	19.97	33.47	23.48	0.78
2	121.22	80.65		94.86	
3	276.22	183.77		216.16	
4	498.55	331.69		390.14	

6 Conclusions

This work investigates the repercussions on the natural frequencies of a concrete beam at the crack openings (physical nonlinearity) and the consequent loss of its stiffness. It also studies the consequences of axial pretension loadings of these same parameters. A concrete beam was analyzed theoretically and numerically, validating the strategy with experimental results for determining the natural frequencies.

The example studied shows that, for both prestressed and reinforced concrete structures, it is relevant to consider both the effects of cracking and prestressing in dynamic analyses because these two effects substantially impact the natural frequencies. An accurate evaluation of these parameters is fundamental in elaborating civil projects.

A decaying abrupt in the natural frequencies values of a structure can mean that it has lost its rigidity. Consequently, an effort redistribution occurs, making it inadequate for users because e there was a violation of the vibration limit state in service. Besides, the prestressed structures present a drastic reduction of the cracks. However, it may also violate normative limits since the system also causes a decrease in natural frequencies.

Therefore, it is possible to conclude that the normative recommendations are deficient by allowing to realize a dynamic evaluation of the concrete buildings without the consideration of the cracking of the sections or the effects related to pretension. For the continuity of this work, it is recommended to use Timoshenko's beam [15], as it presents results with more accuracy since it considers the longitudinal and transversal strains in the general differential equation of the beam

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