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Wave propagation in one-dimensional diatomic metastructure with high-static-low-dynamic stiffness

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Abstract. In this work, we explore wave propagation in a one-dimensional diatomic periodic structure with high-static-low-dynamic stiffness (HSLDS) characteristics, which is a geometric nonlinearity. A diatomic chain consists of two different masses per unit cell, and diatomic periodic structures can present interesting dynamic characteristics, in which waves can attenuate within frequency bands that are called bandgaps. A periodic structure consists fundamentally of identical components, the cells, connected in a way that characteristics of mass, stiffness, and or damping are spatially repeated, and present interesting characteristics for vibration attenuation that are not found in classical structures. These characteristics have been explored for automotive and aerospace applications, among others, as structures with low mass are paramount for these industries, and keeping low vibration levels in a wide frequency range is also desirable.

We use closed-form first-order approximation via perturbation analysis to study wave propagations by dispersion relations of the infinite structure considering the effect of nonlinear terms. We verify the nonlinear bandgap seen via the dispersion relation by comparing it to the transmissibility of a finite structure. We use the dispersion relation to analyse how some parameters can influence the bandgaps, such as the mass ratio between the cell elements and amplitude.

Keywords: HSLDS, Metastructure, Dispersion relation, Bandgap.

1 Introduction

Research on metastructures is attracting increasing attention from many engineering applications, such as civil, automotive and aerospace structures, as they have interesting characteristics such as band gaps and band stops [1]. These characteristics can be manipulated by the macro geometrical arrangement of its fundamental components, or unit cells, in a way that characteristics of mass, stiffness and or damping are spatially repeated and the resulting band gaps are in a desired frequency range [2–4]. According to Chakraborty and Mallik [5] an advantage of metastructure is that the dynamics of these structures can be studied with the analysis of just one cell. According to Mead [6] limiting values of band gaps and band stops can be found by analysing the natural frequencies of free and fixed cells, these frequency ranges can also be found by analysing the transmissibility of a single cell [7].

In order to increase the bandwidth of vibration attenuation in metastructures, nonlinear characteristics have been explored in different ways. Both nonlinear stiffness and damping can affect the dynamic behaviour of such structures [5, 8–10]. High-static-low-dynamic stiffness (HSLDS) is an example of nonlinearity and can be used to attenuate vibrations, as is done in [11–13]. The propagation of acoustic waves can also change due to nonlinear effects [14]. Besides that, metastructures with nonlinear characteristics can have chaotic responses in addition to the periodic ones more commonly observed [15]. To analyse wave propagation, it is possible to use the dispersion relation, as is done in [13], [16] and [17].

In this work, we explore the influence of the mass ratio on the bandgap. We see that it is possible to locate the bandgap of the metastructure by analysing both the transmissibility of just one cell and the dispersion relation of the infinite structure. We also see that the nonlinear term influences the limiting values of bandgap, in addition to influencing quasi-static frequency.

2 Mathematical model

Figure 1 (a) shows a model of the metastructure with axial vibration $F(t)$ that is analysed in this work. Figure 1 (b) shows the geometric change effect in the vertical spring with stiffness coefficient k_a under large displacements. The unit cell with 3 degrees of freedom is shown enclosed within the dotted lines, and is used for the analysis of the finite structure. For the finite structure, we use this cell because it is symmetric. Because, as shown in [18], a periodic structure with symmetrical cells has the same limiting values of bandgap, independent of the number of cells. However, when analyzing the infinite structure, the unit cell shown within the dashed lines is used for convenience, as this has two degrees of freedom. It can be shown that the dispersion relation is the same for the two types of cells.

The metastructure has mass, damping and stiffness coefficients named m_1, m_2, c and k, respectively. The block with mass m_2 is attached to a vertical linear spring with stiffness coefficient k_a. For small displacements of the block in the x direction, there is no influence of the vertical spring. For large displacements, a nonlinear restoring force is present due to geometric arrangement.

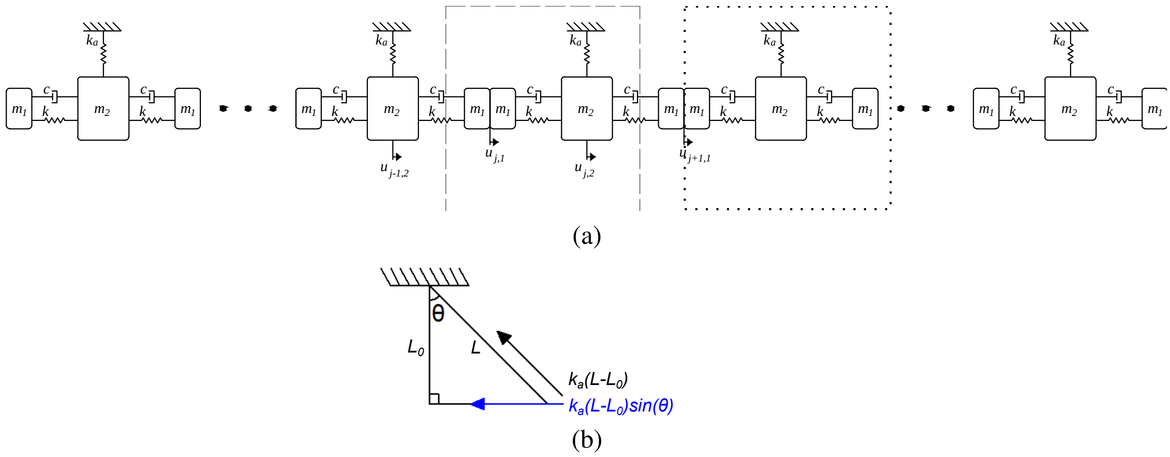


Figure 1. (a) Nonlinear metastructure and (b) geometric change effect in the vertical spring with stiffness coefficient k_a under large displacements.

The vertical spring with stiffness coefficient k_a has an initial length L_0. After being deformed, this spring has a time-varying length of $L = \sqrt{L_0^2 + u_{j,2}^2}$. Since the movement of the metastructure is in the x direction, we must decompose the force in that direction, using:

$$\sin(\theta) = \frac{u_{j,2}}{L} = \frac{u_{j,2}}{\sqrt{L_0^2 + u_{j,2}^2}} \tag{1}$$

substituting eq. (1) in the x component of the force and considering $L = \sqrt{L_0^2 + u_{j,2}^2}$, we have the nonlinear restoring force of the spring with stiffness coefficient k_a, and the approximation of nonlinear term considering $L_0 > 0$ and $0 \leq u_{j,2} \leq 0.5$, given by:

$$k_a u_{j,2} \left(1 - \frac{L_0}{\sqrt{L_0^2 + u_{j,2}^2}}\right) \approx \frac{k_a u_{j,2}^3}{2L_0^2} \tag{2}$$

The dynamical equations can be obtained by applying Newton's second law, giving the general form:

$$M\ddot{x} + C\dot{x} + Kx + G(x) = F(t) \tag{3}$$

in which M, C and K are the mass, damping and stiffness matrices, respectively. x is the displacement vector and $G(x)$ is a vector with nonlinear terms. $F(t)$ is the external force applied to the structure ($F(t) = F_0 \cos(\Omega t)$), in which F_0 is the amplitude and Ω is the frequency of excitation. Note that $F(t)$ is a vector and only its last element is nonzero, to represent an external force applied only at the rightmost mass.

For the cell, the displacement vector is $x = [x_1, x_2, x_3]'$, and the matrices are given by:

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & 2\mu m_1 & 0 \\ 0 & 0 & m_1 \end{bmatrix} \quad C = \begin{bmatrix} c & -c & 0 \\ -c & 2c & -c \\ 0 & -c & c \end{bmatrix} K = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \quad G(x) = \begin{bmatrix} 0 \\ \frac{k_a u_{j,2}^3}{2L_0^2} \\ 0 \end{bmatrix} \tag{4}$$

To solve the linear system it is used the method of mechanical impedance, in which the mechanical impedance Z is found by the following equation: $Z(i\Omega) = -\Omega^2 M + i\Omega C + K$. Frequency response is found by solving X from the following equation: $X = [Z(i\Omega)]^{-1}F(t)$.

The transmissibility is the ratio between the displacement response of the leftmost and rightmost mass, that is, $T = X_1/X_{2n+1}$. To find the frequencies referring to bandgap and bandstop, one must find the expressions to $|T| = 1$, that is given by:

$$\omega_1 = \sqrt{\frac{k}{\mu m_1}} \qquad \omega_2 = \sqrt{\frac{k_1}{m_1}} \qquad \omega_3 = \sqrt{\frac{\mu k + k}{\mu m_1}} \tag{5}$$

2.1 Dispersion relation

We use closed-form first-order approximation via perturbation analysis, as in [16], to study wave propagations by dispersion relations of the infinite structure shown in Fig. 1.

Considering a diatomic chain, as shown in the dashed line in Fig. 1, we can find the following equations of motion:

$$\begin{aligned} 2m_1\ddot{u}_{j,1} + 2ku_{j,1} - ku_{j,2} - ku_{j-1,2} &= 0 \\ 2\mu m_1\ddot{u}_{j,2} + 2ku_{j,2} - ku_{j,1} - ku_{j+1,1} + \epsilon\frac{k_a u_{j,2}^3}{2L_0^2} &= 0 \end{aligned} \tag{6}$$

Multiplying eq. (6) by $\frac{1}{m_1\omega_n^2}$ and considering nondimensional time $\tau = \omega t$, linear natural frequency $\omega_n = \sqrt{k/m_1}$, nondimensional frequency $\bar{\omega} = \omega/\omega_n$ and $\bar{k}_a = \frac{k_a}{m_1\omega_n^2 2L_0^2}$, we have the following equation in matrix form:

$$2\bar{\omega}^2 \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix} \begin{bmatrix} d^2u_{j,1}/d\tau^2 \\ d^2u_{j,2}/d\tau^2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} u_{j,1} \\ u_{j,2} \end{bmatrix} + \begin{bmatrix} -u_{j-1,2} \\ -u_{j+1,1} \end{bmatrix} + \epsilon \begin{bmatrix} 0 \\ \bar{k}_a u_{j,2}^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7}$$

Using the asymptotic expansions:

$$u_j = u_j^{(0)} + \epsilon u_j^{(1)} + 0(\epsilon^2) \qquad \omega = \omega_0 + \epsilon\omega_1 + 0(\epsilon^2) \tag{8}$$

Substituting eq. (8) into eq. (7) and equating the coefficients ϵ^0 and ϵ^1 to zero, we obtain:
Order ϵ^0:

$$2\bar{\omega}_0^2 \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix} \begin{bmatrix} d^2u_{j,1}^{(0)}/d\tau^2 \\ d^2u_{j,2}^{(0)}/d\tau^2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} u_{j,1}^{(0)} \\ u_{j,2}^{(0)} \end{bmatrix} + \begin{bmatrix} -u_{j-1,2}^{(0)} \\ -u_{j+1,1}^{(0)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{9}$$

Order ϵ^1:

$$\begin{aligned} 2\bar{\omega}_0^2 \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix} \begin{bmatrix} d^2u_{j,1}^{(1)}/d\tau^2 \\ d^2u_{j,2}^{(1)}/d\tau^2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} u_{j,1}^{(1)} \\ u_{j,2}^{(1)} \end{bmatrix} + \begin{bmatrix} -u_{j-1,2}^{(1)} \\ -u_{j+1,1}^{(1)} \end{bmatrix} &= \\ -4\bar{\omega}_0\bar{\omega}_1 \begin{bmatrix} d^2u_{j,1}^{(0)}/d\tau^2 \\ d^2u_{j,2}^{(0)}/d\tau^2 \end{bmatrix} - \begin{bmatrix} 0 \\ \bar{k}_a(u_{j,2}^{(0)})^3 \end{bmatrix} \end{aligned} \tag{10}$$

Considering the following solutions:

$$\begin{bmatrix} u_{j,1}^{(0)} \\ u_{j,2}^{(0)} \end{bmatrix} = \begin{bmatrix} (A_1^{(0)}/2)e^{i\kappa a(2j-1)} \\ (A_2^{(0)}/2)e^{i\kappa a(2j)} \end{bmatrix} e^{i\tau} + cc, \quad \begin{bmatrix} u_{j\pm1,1}^{(0)} \\ u_{j\pm1,2}^{(0)} \end{bmatrix} = \begin{bmatrix} (A_1^{(0)}/2)e^{i\kappa a(2(j\pm1)-1)} \\ (A_2^{(0)}/2)e^{i\kappa a(2(j\pm1))} \end{bmatrix} e^{i\tau} + cc \quad (11)$$

Substituting eq. (11) into eq. (9), we obtain two branches for the linear dispersion:

$$\bar{\omega}_0^{opt} = \sqrt{\frac{\sqrt{\mu^2 + (4\cos(a\kappa)^2 - 2)\mu + 1} + \mu + 1}{2\mu}} \quad (12)$$

$$\bar{\omega}_0^{aco} = \sqrt{\frac{-\sqrt{\mu^2 + (4\cos(a\kappa)^2 - 2)\mu + 1} + \mu + 1}{2\mu}} \quad (13)$$

Substituting eq. (12) eq. (13) and into eq. (9), we obtain the amplitude ratio for each mode at ϵ^0 order:

$$\eta_{opt} = \frac{A_2^{(0)}}{A_1^{(0)}} = \frac{2\mu - \sqrt{\mu^2 + (4\cos(a\kappa)^2 - 2)\mu + 1} - \mu - 1}{2\mu\cos(a\kappa)} \quad (14)$$

$$\eta_{aco} = \frac{A_1^{(0)}}{A_2^{(0)}} = \frac{2\mu\cos(a\kappa)}{2\mu + \sqrt{\mu^2 + (4\cos(a\kappa)^2 - 2)\mu + 1} - \mu - 1} \quad (15)$$

Substituting eq. (11) into the second component of eq. (10), we obtain:

$$2\mu\bar{\omega}_0^2 \frac{d^2 u_{j,2}^{(1)}}{d\tau^2} + 2u_{j,2}^{(1)} - u_{j,1}^{(1)} - u_{j+1,1}^{(1)} = d_1 e^{2i\kappa ja} e^{i\tau} + d_3 e^{6i\kappa ja} e^{3i\tau} + cc \quad (16)$$

The term to the left of the equality in eq. (16) is similar to eq. (9). Therefore, the terms that are secular and must be eliminated are those that multiply $e^{i\tau}$ in eq.(16) and are given by:

$$d_1 = 2A_2^{(0)}\bar{\omega}_0\bar{\omega}_1 - \frac{3(A_2^{(0)})^2\bar{A}_2^{(0)}\bar{k}_a}{8} \quad (17)$$

Equating d_1 term shown in eq. (17) to zero, we obtain the first-order frequency correction:

$$\bar{\omega}_1 = \frac{3A_2^{(0)}\bar{A}_2^{(0)}\bar{k}_a}{16\bar{\omega}_0} \quad (18)$$

Substituting eq. (12) into eq. (18), and substituting this result and eq. (12) into second eq. (8), we obtain the optical branch:

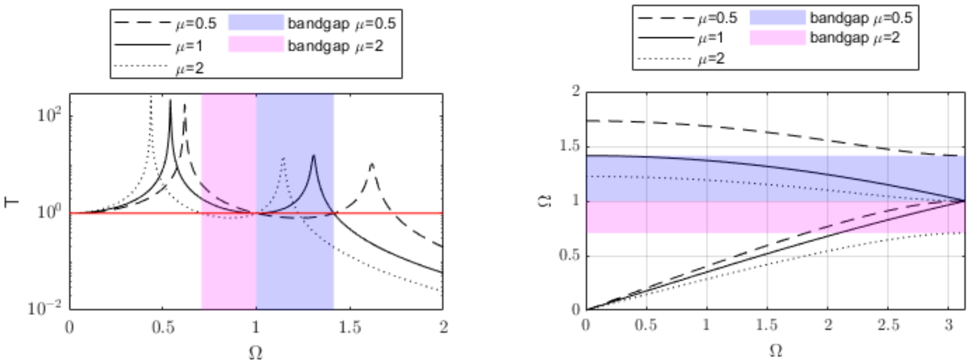
$$\bar{\omega}_{opt} = \frac{2^{7/2}\sqrt{\mu^2 + (4\cos(a\kappa)^2 - 2)\mu + 1} + (3\sqrt{2} \mid A_2^{(0)} \mid^2 \epsilon\bar{k}_a + 2^{7/2})\mu + 2^{7/2}}{16\mu\sqrt{\frac{\sqrt{\mu^2 + (4\cos(a\kappa)^2 - 2)\mu + 1} + \mu + 1}{\mu}}} \quad (19)$$

Substituting eq. (13) into eq. (18), and substituting this result and eq. (13) into second eq. (8), we obtain the acoustic branch:

$$\bar{\omega}_{aco} = -\frac{2^{7/2}\sqrt{\mu^2 + (4\cos(a\kappa)^2 - 2)\mu + 1} + (-3\sqrt{2} \mid A_2^{(0)} \mid^2 \epsilon\bar{k}_a - 2^{7/2})\mu - 2^{7/2}}{16\mu\sqrt{-\frac{\sqrt{\mu^2 + (4\cos(a\kappa)^2 - 2)\mu + 1} - \mu - 1}{\mu}}} \quad (20)$$

3 Dynamical response of linear metastructure

Figure 2 shows the transmissibility of one cell and the linear relation dispersion of infinite structure, bandgaps are represented by magenta and blue regions, in which the limiting values are that found by eq. (5). In both transmissibility and dispersion relation we use three values of mass ratio: $\mu = 0.5$ (dashed line), $\mu = 1$ (continuous line) and $\mu = 2$ (dotted line). It is possible to note that for $\mu = 1$ there is no bandgap.



(a) (b)

Figure 2. Transmissibility (a) and dispersion curve (b) of linear metastructure with three values of mass ratio $\mu = 0.5$ (dashed line), $\mu = 1$ (continuous line) and $\mu = 2$ (dotted line).

If $\mu < 1$, we have the bandgap starting at ω_2 and ending at ω_1, if $\mu > 1$ we have the opposite, the bandgap starting at ω_1 and ending at ω_2 and the closer μ is to 1, smaller the bandgap range.

4 Dynamical response of nonlinear metastructure

In the nonlinear metastructure, we analysed the influence of nonlinearity on the bandgap. To do this analysis, we use the stiffness coefficient $k_a = 5 \times 10^{-3}$ and vary the amplitude $A_2^{(0)}$. In the analysis of the linear metastructure, we saw that for $\mu = 1$, there is no bandgap, so we analysed the nonlinear metastructure with $\mu = 0.5$ and $\mu = 2$.

Figure 3 shows the nonlinear dispersion relation with $\mu = 0.5$ and Figure 3 shows with $\mu = 2$. (a) shows the dispersion relation for κ from 0 to π, (b) a zoom of optical branch at κ close to 0, (c) a zoom of optical branch at κ close to π, (d) a zoom of acoustic branch at κ close to 0, and (e) a zoom to κ close to π.

Figure 3 (d) shows the influence of nonlinearity on the quasi-static frequency, in which, the increase in amplitude generates an increase in the bandgap region at this frequency range. Figures 3 (c) and (e) shows the influence of nonlinearity in the bandgap region seen in the linear analysis. Here, we can see that increasing the amplitude causes a shift to higher frequencies at the beginning and end of the bandgap.

Figure 4 shows an influence similar to the one seen in Fig. 3. That is, when analyzing the nonlinear metastructure with $\mu = 2$, we also have the influence of nonlinearity in the bandgap of the quasi-static region and the bandgap region seen in the linear analysis. There is only one difference in the quasi-static region. We can see that for this region, there is a larger bandgap for $\mu = 2$.

5 Conclusions

In this work, we explore wave propagation in a one-dimensional diatomic periodic structure with high-static-low-dynamic stiffness characteristics, which is a geometric nonlinearity. From the results shown in section 3, we see that it is possible to find the bandgap by analysing the transmissibility of just one cell, in which these regions could also be seen in the dispersion relation. For there to be a bandgap, the mass ratio must be different from 1, if $\mu < 1$, we have the bandgap starting at ω_2 and ending at ω_1, if $\mu > 1$ we have the opposite, the bandgap starting at ω_1 and ending at ω_2 and the closer μ is to 1, smaller the bandgap range. For the nonlinear metastructure, we can observe the attenuation in the quasi-static region, and also an influence on the bandgap limiting frequencies, both for $\mu = 0.5$ and for $\mu = 2$. The increase in amplitude $A_2^{(0)}$ increases the influence in the quasi-static region and the bandgap.

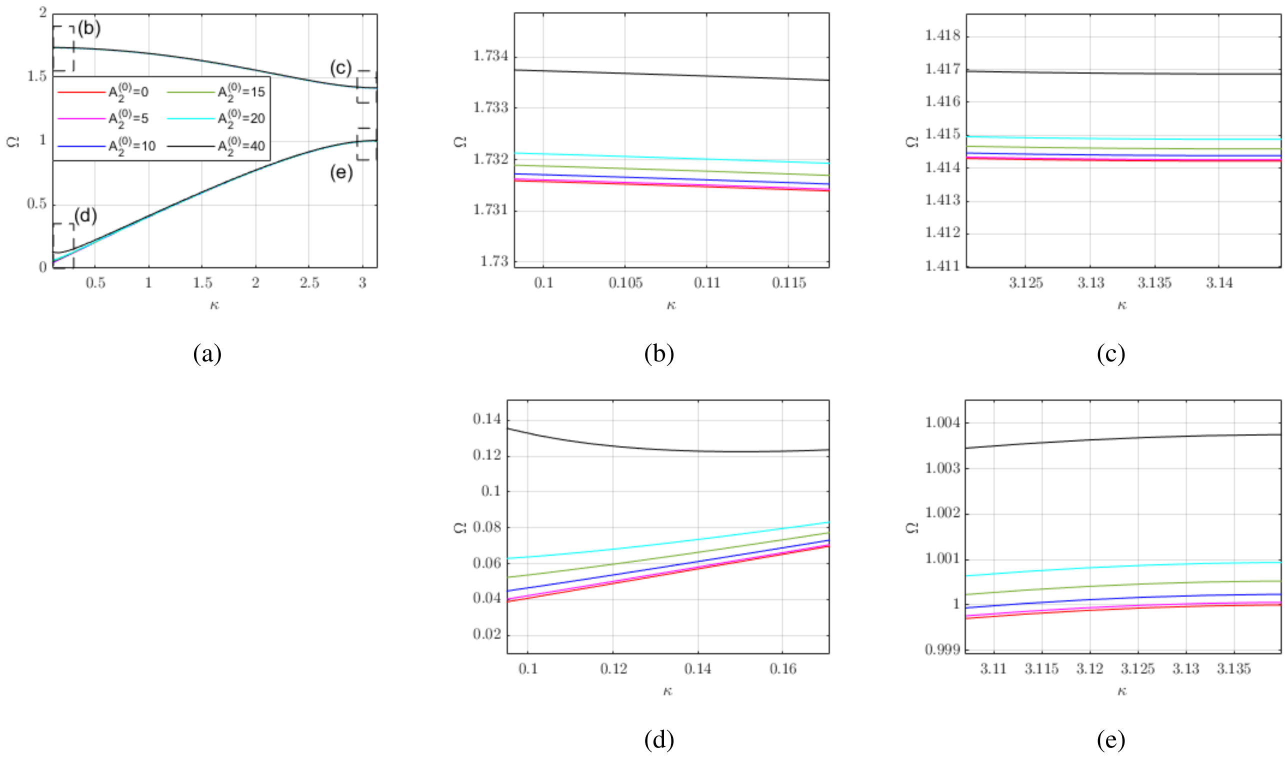


Figure 3. Nonlinear dispersion relation with $\mu = 0.5$ to some values of $A_2^{(0)}$ $(0, 5, 10, 15, 20$ and $40)$. (a) dispersion relation for κ from 0 to π, (b) zoom of optical branch at $\kappa = 0$, (c) zoom of optical branch at $\kappa = \pi$, (d) zoom of acoustic branch at $\kappa = 0$ and (e) zoom of acoustic branch at $\kappa = \pi$.

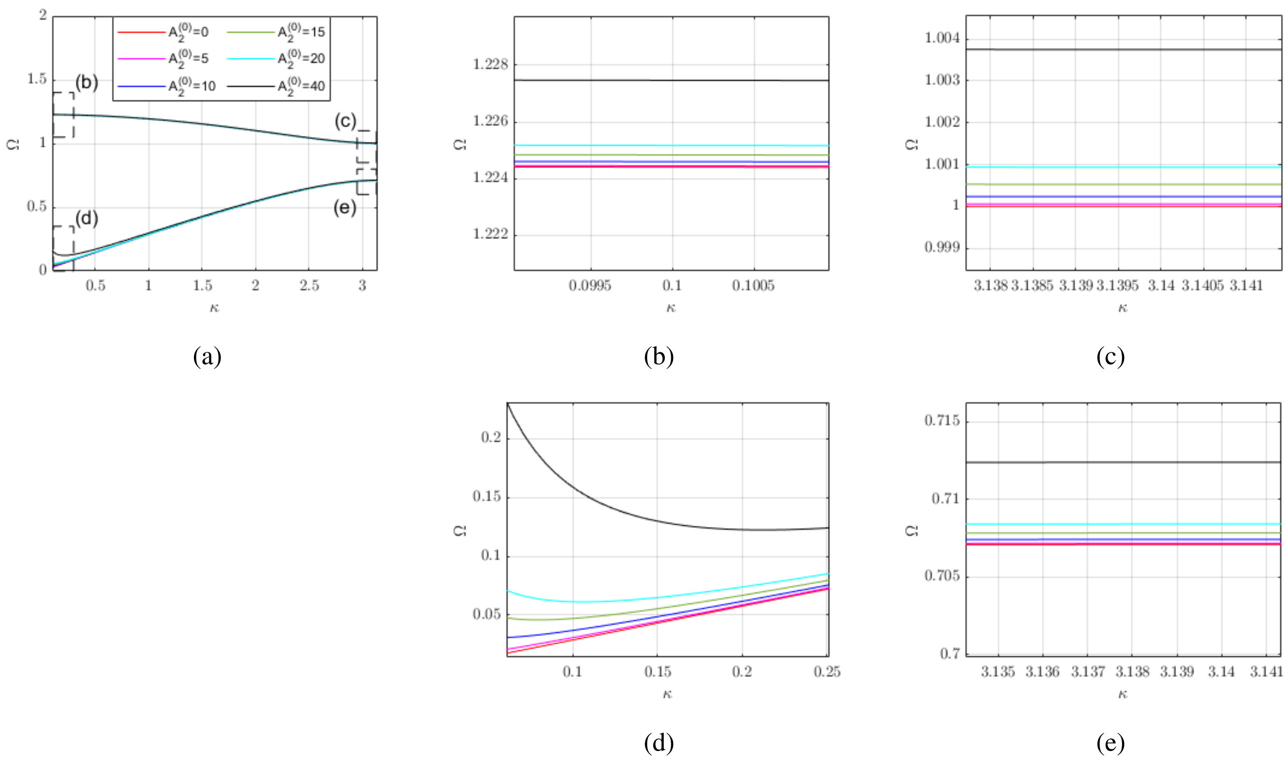


Figure 4. Nonlinear dispersion relation with $\mu = 2$ to some values of $A_2^{(0)}$ $(0, 5, 10, 15, 20$ and $40)$. (a) dispersion relation for κ from 0 to π, (b) zoom of optical branch at $\kappa = 0$, (c) zoom of optical branch at $\kappa = \pi$, (d) zoom of acoustic branch at $\kappa = 0$ and (e) zoom of acoustic branch at $\kappa = \pi$.

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