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Numerical solution of axisymmetric shells under bilateral contact constraints

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Abstract. In structural analysis, the possible geometrical configurations of the structure are fundamental for the choice of the coordinate system to be used and, consequently, for the determination of solutions. The structural problem description through an appropriate coordinate system allows us to visualize aspects that would not necessarily be observed in another coordinate system. In this sense, the correct specification of the coordinate system is important and necessary. The most general coordinate system, of which all others are particular cases, is the general curvilinear coordinate system. Simultaneously, differential equations are widely used to model problems in structural engineering. In the solution of a differential equation, the finite difference method plays an important role, promoting the discretization of space by a mesh of discrete points, with the unknown functions and their derivatives being replaced by approximations at the grid points through difference quotients. In particular, the analysis of contact problems between a structure and a deformable foundation is an essential task in civil engineering, with crucial use in different support systems. Undoubtedly, thin axisymmetric shells are structural elements widely used in structural engineering, especially when interacting with elastic or inelastic means. The main objective of this work is to develop a computational tool for the study and analysis of problems under bilateral contact constraints, involving axisymmetric shells and elastic means, using the approximations derived from Sander's theory for slender shells. General equations applicable to any coordinate systems are developed. As an application example, a slender cylindrical shell supported on soil (elastic foundation) is used.

Keywords: Axisymmetric Shells, Soil, finite difference method, Contact Constraints, Coordinate Systems

1 Introduction

This paper presents a numerical methodology for obtaining the structural behavior of axisymmetric shells with contact constraints. Analysis of contact problems between a construction and a deformable foundation is an essential task in civil engineering, with crucial use in different structural systems. Fundamentally, the structures are built in soil/rock formations, and the loads that operate in the structures are transmitted to the support site through the foundation elements, causing deformation in the support of the structure which, in turn, modifies the tensions in the whole. In this sense, the analysis of the structure-foundation systems lies in soil/rock response to the applied loads, in the contact region, with the inclusion of the foundation reaction in the differential equation associated with the problems involving beams, plates or peels [1, 2].

Axisymmetric structures constitute a part of substantial engineering constructs. Such conformations comprise, for example, conductive tubes, reservoir, cooling towers, cylindrical containment compositions, chimneys and roofs. Essentially, axisymmetric geometries have an axial symmetry axis, regardless of the circumferential coordinate. A structure with axisymmetric configuration allows significant simplifications to be performed by investigating the deformation that occurs on the reference surface [3].

Shell is a three-dimensional object with one of the three dimensions much smaller than the other two. The geometry of a shell is specified from the reference surface, thickness and contours. Generally, the shell thickness is a variable amount at each point. However, in this work, shells with constant thickness are used as the study structure. In this sense, the middle surface is taken as the reference surface because, in addition to determining the shape of the shell, it specifies, at all points of the structure, the states of strain and stress.

In structural analysis, the possible configurations of the structure are fundamental for the choice of the coordinate system to be used and, consequently, for the determination of possible solutions. The description of a problem based on an adequate coordinate system makes it possible to visualize aspects that would not necessarily be observed in another coordinate system. In this sense, the correct specification of the coordinate system is important and necessary. The most general coordinate system, of which all others are particular cases, is the general curvilinear coordinate system, described by the coordinates Θ^1, Θ^2 and Θ^3. In this system, all coordinate lines are curved – called coordinate curves – and can form any angle with each other.

The interaction action between structures and elastic foundations is of fundamental importance in foundation design and it has always attracted the attention of both researchers and engineers. The mathematical model used to represent the foundation depends on the application considered, with the determination of the contact pressure between the parts being the main difficulty of the analysis in the interaction between the soil and the structure. Among the methods commonly applied to investigate structural systems, there are those that adopt numerical modeling, with emphasis on those that consider the foundation an elastic medium, especially models that use discrete springs. The study of the contact between axisymmetric shells and elastic base is based on the way in which the interaction between these elements occurs. In this sense, the permanent fixation of the parts, without loss of contact, characterizes the problem of bilateral contact (PCB) (Fig. 1) [4, 5].

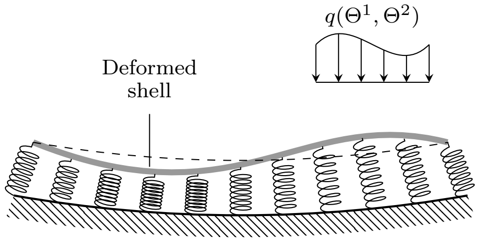


Figure 1. Problem of bilateral contact.

Problems involving axisymmetric shells supported by elastic bases can be solved using numerical methods associated with computational methods. In this sense, the finite difference method (FDM) provides results close to those considered real [6]. This work is based on the finite difference formulation to approximate the derivatives present in the equations obtained in Section 2. The process of discretization of space takes into account the organization of the chosen points, thus defining the mesh of finite differences. Depending on the arrangement imposed, the mesh can be composed of structures of different formats, the most common being squares, rectangles and triangles, always in line with each specific problem and methodology used.

In this article, the general mathematical structure is presented. Bilateral contact problems between shells and elastic bases are solved using the finite difference method. Cylindrical shells are used as an application of general equations.

2 Theoretical Formulation

2.1 Thin Axisymmetric Shells

The development of thin axisymmetric shells theory is based on the understanding of surface theory, since the shell geometry is specified from the reference surface. Therefore, the fundamental forms are extremely important and useful in determining the metric properties of a surface, such as line element, area element, normal curvature, gaussian curvature, and mean curvature. The first fundamental form allows the calculation of metric quantities and the second fundamental form establishes how a surface is curved [7].

The points that belong to the surface are expressed as functions of the curvilinear coordinates Θ^1 and Θ^2. The union of these curves coordinate generates a surface, where each point is determined by the intersection of these curves coordinate. From the two fundamental forms, the geometric description of a surface with respect to curves Θ^1 and Θ^2 is made by the two Lamé parameters, h_1 and h_2, determined by the module of vectors tangent to curves coordinate, and the two principal radii of curvatures, R_1 and R_2.

The shell geometry is represented by its reference surface and described by a system of orthogonal curvilinear coordinates Θ^1, Θ^2 and ζ. The quantities of interest are determined for reference surface points as functions of the curvilinear coordinates Θ^1 and Θ^2, and, therefore, defined for any other surface of the shell, from a small distance ζ (Fig. 2). Thus, deformations of any point can be expressed in terms of the surface parameters in which it is found that, in turn, are determined from the reference surface. A shell is said to be thin when the ratio between the thickness t and the radius of curvature R can be neglected when compared to unity.

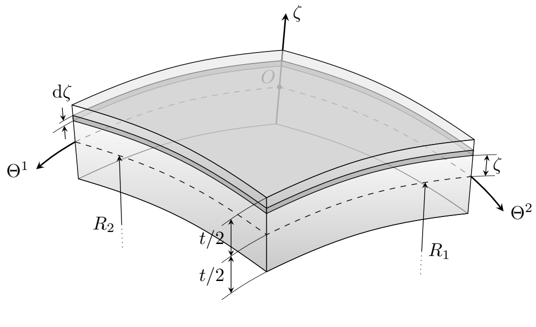


Figure 2. Quantities of interest in the shell element.

The theory of the thin shell is obtained from the three dimensional elasticity, with a few simplifying assumptions, known as Love's first approximation. These assumptions simplify the three-dimensional relations, that are converted into two-dimensional ones. In a system of orthogonal curvilinear coordinate, the deformations ε_{ij} at any point in the thin shell are:

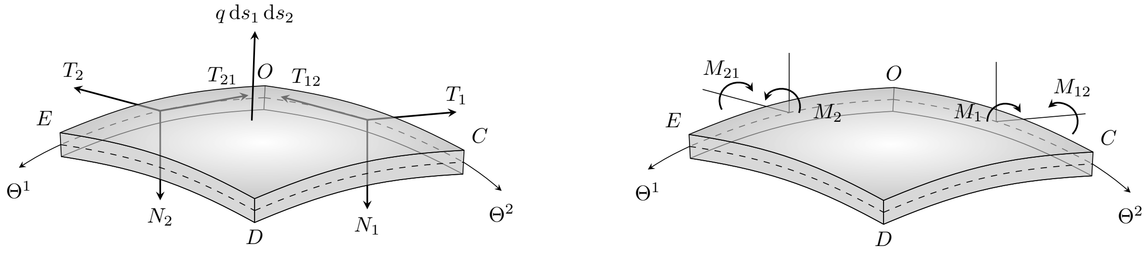
$$
\begin{aligned}
&\varepsilon_{11} = \varepsilon_{11}^0 + \zeta\,\chi_{11}, \quad \varepsilon_{22} = \varepsilon_{22}^0 + \zeta\,\chi_{22}, \quad \varepsilon_{12} = \varepsilon_{21} = \varepsilon_{12}^0 + 2\zeta\,\chi_{12}, \\
&\varepsilon_{33} = 0, \qquad \varepsilon_{13} = \varepsilon_{31} = 0, \qquad \varepsilon_{23} = \varepsilon_{32} = 0,
\end{aligned}
\tag{1}
$$

where ε_{ij}^0 represent the change in the length of the shell elements on the reference surface, while the terms χ_{ij} constitute the changes in curvatures and the twist of the reference surface. Indexes 1, 2 and 3, indicate the directions to Θ^1, Θ^2 and ζ, respectively. Using the approach of Sanders [8], the terms ε_{ij}^0 (Eqs. 1) are:

$$
\varepsilon_{11}^0 = \varepsilon_{10} + \frac{1}{2}\left[(\beta_n)^2 + (\beta_1)^2\right], \quad \varepsilon_{22}^0 = \varepsilon_{20} + \frac{1}{2}\left[(\beta_n)^2 + (\beta_2)^2\right], \quad \varepsilon_{12}^0 = \omega_1 + \omega_2 + \beta_1\beta_2, \tag{2}
$$

where β_i and ω_i represents rotations in the reference surface.

Obtaining the static equilibrium equations of a thin shell takes into account a shell element limited by four normal cuts to the reference surface. By establishing such equations, it should be considered not only the dimensional changes of the element, but also the fact that, due to the curvature of the curves Θ^1 and Θ^2, the forces acting on the element faces may have components in different directions from those cited nominally. The analysis of static equilibrium conditions takes into account the action of a set of forces on the investigated shell and the balance consideration in any parts of the structure. The resultant vector of all internal and external forces acting on an element of the shell must be vanish, just like the resultant moment of all forces. The six forces per unit of length T_1, T_{12}, N_1, T_2, T_{21}, N_2, the four moments per unit of length M_1, M_{12}, M_2, M_{21}, and the external force q, acting on a unit area of the middle surface, characterize completion the state of tension and are shown in Fig. 3.



(a) Forces in the shell element.

(b) Moments in the shell element.

Figure 3. Forces and moments in the shell element.

From equilibrium conditions for forces and moments, it is found:

$$\frac{1}{h_1 h_2}[\partial_1(h_2 T_1) + \partial_2(h_1 T_{21}) + \partial_2 h_1 T_{12} - \partial_1 h_2 T_2] + \frac{N_1}{R_1} + q_1 + r_{b_1} = 0, \tag{3}$$

$$\frac{1}{h_1 h_2}[\partial_1(h_2 T_{12}) + \partial_2(h_1 T_2) + \partial_1 h_2 T_{21} - \partial_2 h_1 T_1] + \frac{N_2}{R_2} + q_2 + r_{b_2} = 0, \tag{4}$$

$$\frac{1}{h_1 h_2}[\partial_1(h_2 N_1) + \partial_2(h_1 N_2)] - \frac{T_1}{R_1} - \frac{T_2}{R_2} + q_3 + r_{b_3} = 0, \tag{5}$$

$$\frac{1}{h_1 h_2}[\partial_1(h_2 M_1) + \partial_2(h_1 M_{21}) + \partial_2 h_1 M_{12} - \partial_1 h_2 M_2] - N_1 = 0, \tag{6}$$

$$\frac{1}{h_1 h_2}[\partial_1(h_2 M_{12}) + \partial_2(h_1 M_2) + \partial_1 h_2 M_{21} - \partial_2 h_1 M_1] - N_2 = 0, \tag{7}$$

$$T_{12} - T_{21} + \frac{M_{12}}{R_1} - \frac{M_{21}}{R_2} = 0. \tag{8}$$

where r_{b_i} is the pressure exerted by the elastic foundation.

Assuming that the material obeys Hooke's law, isotropic, and following Love's first approximation, the stress-strain relations for a three-dimensional thin axisymmetric shells are:

$$\sigma_{11} = \frac{E}{1-\nu^2}(\varepsilon_{11} + \nu\varepsilon_{22}), \qquad \sigma_{22} = \frac{E}{1-\nu^2}(\varepsilon_{22} + \nu\varepsilon_{11}), \qquad \sigma_{12} = \sigma_{21} = \frac{E}{2(1+\nu)}\varepsilon_{12}, \tag{9}$$

where E, G, and ν are the elastic modulus, shear modulus, and Poisson's ratio of the shell material, respectively.

2.2 Elastic Foundations

The mathematical model used to represent the foundation depends on the application considered, being the determination of contact pressure between the parts the main difficulty of analysis in the interaction between the ground and the structure. Among the methods commonly applied for the investigation of the structural system are those who use numerical modeling, especially those who consider the foundation an elastic medium, especially the models that use discrete springs. In such approximations, the behavior of the elastic foundation is modeled using (i) a parameter, where soil-structure interaction is represented by independent discrete springs that associate its rigidity with the properties of elastic foundation material, (ii) two parameters, in which the interaction between the springs is added [1].

The simplest mechanical model was proposed by Winkler in 1867 [9]. In this representation, the pressure exerted by the base at a given point is directly proportional to deflection on site. Mathematically:

$$r_b(\Theta^1, \Theta^2) = k_B w_b(\Theta^1, \Theta^2), \tag{10}$$

where k_B is the foundation modulus and $w_b(\Theta^1, \Theta^2)$ is the deflection. As soil may exhibit considerable interaction action among its elements, the Winkler springs have inherited deficiency in simulating such behavior. Pasternak assumes the existence of shear interactions between the spring elements [10]. Mathematically:

$$r_b(\Theta^1, \Theta^2) = k_B w_b(\Theta^1, \Theta^2) - k_M \, \nabla^2 w_b(\Theta^1, \Theta^2), \tag{11}$$

where the second term on the right-hand side is the effect of the shear interactions of the vertical elements.

2.3 Spatial Discretization

The partial differential equations for shell element (Eqs. (3) - (8)) are solved here by the finite difference method. This method promotes the discretization of space by a mesh of discrete points, with unknowns and their derivatives being replaced by approximations at the points of the grid through difference quotients. Characteristically, for each nodal point, a set of algebraic equations is obtained.

The process of discretization of space takes into account the organization of the stitched points, thus defining the mesh of finite differences. Depending on the arrangement imposed, the mesh can be composed of structures of different shapes, the most common squares, rectangles and triangles, always depending on each specific problem and methodology used. Figure 4 schematizes the finite difference meshes in the curvilinear coordinate system for the pivotal point (i, j).

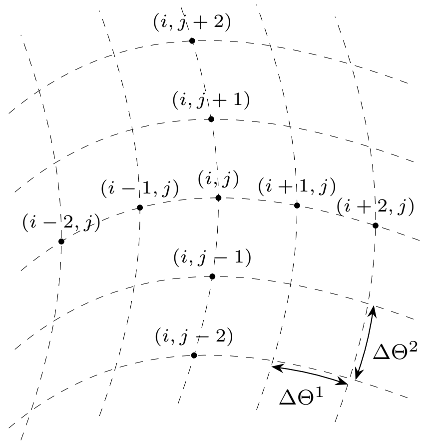


Figure 4. Finite difference meshes in the curvilinear coordinate system for the pivotal point (i, j).

3 Applications

3.1 Solution Strategies

The numerical strategies used in this work for the approximate solution of bilateral contact problems have as main characteristics:

i. The use of the finite difference method (FDM), which replaces the original domain of the bodies (structure and elastic base) and their respective contours with a mesh. As a consequence, the algebraic equation system that governs the PCB is reached;
ii. After this system discretization, the solution of the problem can be directly achieved by this system of algebraic equations;
iii. The computer program used for numerical simulation was made in the Fortran 90 language.

3.2 Example

As an example of application, a thin cylindrical panel is used, since cylindrical panel is a particular case of thin cylindrical shells (thin axisymmetric shell), with extensive radius and opening angles of less than 30°. For that, the cylindrical curvilinear coordinate system is used to write strain-stress relationships and equilibrium equations. To obtain the fundamentals equations for the study a thin cylindrical shell, according to the scope of this work, it is convenient to introduce the following definitions (Fig. 5):

$$\Theta^1 \equiv x, \qquad \Theta^2 \equiv \theta, \qquad \Theta^3 \equiv R. \tag{12}$$

With this, the correspondence between the general curvilinear coordinates and the cylindrical coordinate system is made by:

$$1 \to x, \qquad 2 \to \theta, \qquad 3 \to R. \tag{13}$$

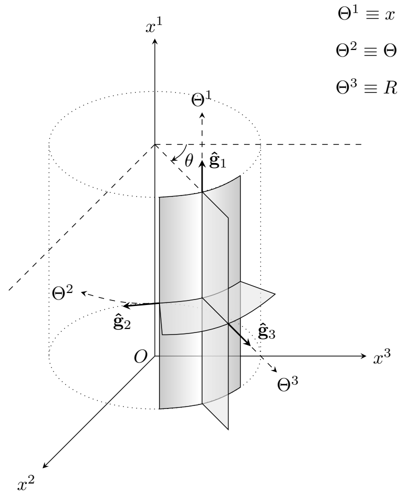


Figure 5. Cylindrical curvilinear coordinate system. The unit vectors $\hat{\mathbf{g}}_1$, $\hat{\mathbf{g}}_2$ and $\hat{\mathbf{g}}_3$ indicate the coordinated directions.

The numerical example consists of a rectangular panel of $7.2 \times 14.4\,\mathrm{m}^2$ subject to a distributed loading of $2.394 \times 10^4\,\mathrm{N/m}^2$ [11]. Some adaptations had to be done to bring the problem analyzed in the literature. Firstly, a $1000\,\mathrm{m}$ radius of the cylindrical panel was considered to resemble the flat plate. Second, the use of simple supports on smaller sides was adopted, as this program does not allow the use of free edges on the four sides. The base used by Straughan follows Pasternak's model, with $k_B = 1.2993\,\mathrm{MN/m}^3$ e $k_M = 2.54652\,\mathrm{MN/m}$. In this case, was used $E = 20.684\,\mathrm{GPa}$, $\nu = 0.0$ and $t = 0.15\,\mathrm{m}$.

Figure 6 presents the radial displacement found at points located on a line passing in the middle of the panel, parallel to the smaller sides (simply supported). To model the structural system (shell and foundation) two meshes with 13×25 and 21×41 discrete points each were used. The results show a difference between these meshes in the computational domain. As observed, all grids give the same behavior of results obtained by Straughan [11], with better approach to the mesh with 13×25 nodes. There is a good approximation of the deformed configurations, mainly in the center of the structure, with the differences observed from the rigidity gain provided by the differences in contour conditions (free to simply supported). In addition, can be observed in this example that the adoption of the Winkler model – using only the first k_B parameter –, besides offering less opposition to the displacement, unlike the two parameters model. It also provided little stiffness to the shell-base system (it is observed that the graph's displacement line showed a very smooth curvature, almost straight). This behavior is expected, even in the case of this problem, which has a relationship between k_B and k_M of reasonable size ($k_B/k_M = 0.51$).

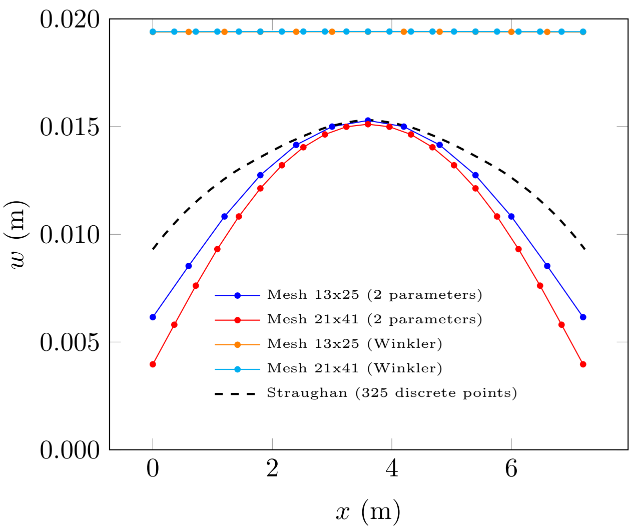


Figure 6. Radial displacement in a smooth curvature panel on an elastic base.

4 Conclusions

This work has developed a computational tool for the study and analysis of problems involving cylindrical panels with bilateral contact restrictions imposed by elastic bases. It was also the object of study the use of the FDM as a tool for discretion of the continuum, transforming the differential equations of both the shell and the basis models into algebraic equations. The finite difference method showed efficiency in discretization of the structural problem, presenting ease of implementation of both cylindrical shell theory and also elastic basic models, characterized as an alternative viable to other numerical methods. For the example studied, the main models of elastic bases were used, highlighted the two most common types: one or two parameters.

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