Elastic-Viscoplastic analysis of Reissner's plates by the Boundary Element Method

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Abstract. This work presents a formulation for elastic-viscoplastic analysis of plate bending by the Boundary Element Method (BEM). It is employed Reissner's theory, in which transverse shear strains are considered and so, it holds for thin and thick plates. Basic formulation of Reissner's plate bending theory is presented, with the consideration of physical nonlinearity. The related integral equations are shown for displacements at internal and boundary points and also for moments and shear resultants at internal points. The theory for considering viscoplasticity is presented, as well as the procedures for the solution of these equations by the BEM. This process offers an alternative method of solution for elasto-plastic problems, when steady-state condition is reached. For the numerical implementation, quadratic boundary elements of linear geometry and constant internal cells, also of linear geometry, are employed. These cells are only necessary where the existence of inelastic strains is expected. Numerical examples are presented and the obtained results are compared with solutions found in literature.

Keywords: Viscoplasticity, Reissner's plates, Boundary element method

1 Introduction

The Boundary Element Method (BEM) has been used to solve several kinds of nonlinear engineering problems, however there are still few works in literature considering viscoplascity related to plate bending.

In this context, Morjaria and Mukherjee [1] developed a formulation for plate bending inelastic analysis considering a viscoplastic model for Kirchhoff's plate theory, in which simply supported and clamped plates were analyzed considering the transverse loading uniformly distributed and varying linearly with time. Providakis [2] used a methodology that combinates BEM and Finite Element Method (FEM) to obtain the integral formulation to analyze plates with arbitrary boundary conditions to general lateral loading history. The BEM in its direct form with the nonhomogeneous biharmonic equation for time-dependent inelastic analysis it was used and the plate material is modelled as elastic-viscoplastic. These mentioned works are based on classical plate bending theory, and therefore, are only applied to thin plates.

The present work presents an elastic-viscoplastic formulation for plate bending analysis using the boundary element method, considering the plate bending theory of Reissner [3], which is applied for both thick and thin plates. A procedure similar to that adopted by Telles and Brebbia [4] and Telles [5] for two-dimensional and three-dimensional elastic-viscoplastic problems was considered. Perzyna's approach given in Ref. [6] is considered and can also be used to simulate a pure elastoplastic problem. A simple Euler one step procedure is used to obtain the time dependent solution. The criteria of von Mises and of Tresca are adopted.

For the numerical implementation, quadratic boundary elements of linear geometry and constant internal cells, also of linear geometry, are employed. These cells are only necessary where the existence of inelastic strains is expected.

The cartesian tensor notation is used throughout the text, with Greek indices varying from 1 to 2 and Latin ones from 1 to 3. In addition, the dot over some variables indicates time derivative.

2 Governing equations for nonlinear material Reissner's plates

Let h be a constant thickness plate with homogeneous, isotropic and nonlinear material, subject to a transverse load q per unit area. Cartesian axis x_α is considered in the plate middle surface and x_3 in the transverse direction.

The transverse displacement rate $\dot{u}_3$ and rotation rates $\dot{u}_\alpha$ are considered in this plate analysis.

Admitting that the plate has inelastic strains only due to bending, the total bending strain rate $\dot{\chi}_{\alpha\beta}$ and the total shear strain rate $\dot{\psi}_\alpha$ are given by the following expressions:

$$\dot{\chi}_{\alpha\beta} = \frac{1}{2}\left(\dot{u}_{\alpha,\beta} + \dot{u}_{\beta,\alpha}\right) = \dot{\chi}^{e}_{\alpha\beta} + \dot{\chi}^{a}_{\alpha\beta} \tag{1}$$

$$\dot{\psi}_\alpha = \dot{u}_\alpha + \dot{u}_{3,\alpha} = \dot{\psi}^{e}_{\alpha} \tag{2}$$

where $\dot{\chi}^{e}_{\alpha\beta}$ and $\dot{\psi}^{e}_{\alpha}$ are elastic parcels and $\dot{\chi}^{a}_{\alpha\beta}$ is the inelastic parcel, which can be due to viscoplasticity, $\dot{\chi}^{\mathrm{p}}_{\alpha\beta}$.

The expressions for moment rates and transverse shear force rates are given in the following form:

$$\dot{M}_{\alpha\beta} = \frac{D(1-\nu)}{2}\left[2\dot{\chi}_{\alpha\beta} + \frac{2\nu}{1-\nu}\dot{\chi}_{\gamma\gamma}\delta_{\alpha\beta}\right] + \frac{\nu\dot{q}}{(1-\nu)\lambda^2}\delta_{\alpha\beta} - \dot{M}^{a}_{\alpha\beta} \tag{3}$$

$$\dot{Q}_\alpha = \frac{D(1-\nu)\lambda^2}{2}\dot{\psi}_\alpha \tag{4}$$

in which $\delta_{\alpha\beta}$ represents the Kronecker delta, ν is Poisson's ratio, $\lambda = \sqrt{10}/h$ is the characteristic constant of Reissner's equations and $D = Eh^3/12(1-\nu^2)$ is the plate bending rigidity, with E being Young's modulus. The term $\dot{M}^{a}_{\alpha\beta}$ expresses the components of inelastic moment rates, defined in initial stress formulation as:

$$\dot{M}^{a}_{\alpha\beta} = \frac{D(1-\nu)}{2}\left[2\dot{\chi}^{a}_{\alpha\beta} + \frac{2\nu}{1-\nu}\dot{\chi}^{a}_{\gamma\gamma}\delta_{\alpha\beta}\right] \tag{5}$$

Generalized traction rates $\dot{p}_\alpha$ and $\dot{p}_3$ are expressed by:

$$\dot{p}_\alpha = \dot{M}_{\alpha\beta}n_\beta \tag{6}$$

$$\dot{p}_3 = \dot{Q}_\alpha n_\beta \tag{7}$$

where n_β represents the direction cosines of the outward normal to the boundary.

3 Elastic-viscoplastic constitutive equations

Considering Perzynas's model (Ref. [6]) and using a procedure similar to that employed by Telles and Brebbia [4] and Telles [5] for two- and three-dimensional problems, the constitutive equations for plate bending by Reissner's theory with the consideration of an elastic-viscoplastic model are obtained.

It is considered a yield function, F, that does not differ from the corresponding yield condition for the inviscid theory of plasticity. expressed in terms of moments $M_{\alpha\beta}$ and a hardening parameter κ, as

$$F\left(M_{\alpha\beta}, \kappa\right) = f\left(M_{\alpha\beta}\right) - \Psi(\kappa) = M_e - M_0 = 0 \tag{8}$$

in which M_e is the equivalent moment and can be calculated by different criteria presented in literature. In the present work, von Mises' and Tresca's yield criteria are adopted.

When the equivalent moment M_e reaches the uniaxial yield moment M_0, the entire cross section plastifies. So, considering that σ_y is the uniaxial yield stress, one has the related initial uniaxial yield moment $M_y = \sigma_y h^2/4$.

For a more general case, where linear hardening is considered, the uniaxial yield moment can be written as:

$$M_0 = M_y + H'\chi^{p}_{e} \tag{9}$$

in which $H' = dM/d\chi^{p}_{e}$ represents a constant slope of the strain hardening portion of the stress-strain curve after the removal of the elastic strain component and χ^{p}_{e} is the current effective viscoplastic strain.

An increase of equivalent viscoplastic strain rate $\dot{\chi}^{p}_{e}$ causes an increase in the plastic strain energy, so:

$$M_e\dot{\chi}^{p}_{e} = M_{\alpha\beta}\dot{\chi}^{p}_{\alpha\beta} = \dot{\kappa} \tag{10}$$

Through the normality principle of Perzyna [6], the viscoplastic strain rate can be given by:

$$\dot{\chi}^{p}_{\alpha\beta} = \gamma\left\langle \Phi\left(\frac{F}{\Psi}\right)\right\rangle \frac{\partial F}{\partial M_{\alpha\beta}} \tag{11}$$

where γ is the fluidity parameter which can be function of time, temperature, etc and

$$\langle \Phi\left(\frac{F}{\Psi}\right)\rangle \begin{cases} 0 \quad \text{for } F \leq 0 & \text{Elatic behavior} \\ \Phi\left(\frac{F}{\Psi}\right) \quad \text{for } F > 0 & \text{Viscoplastic behavior} \end{cases} \tag{12}$$

Perzyna [6] proposes different types of Φ and, in the present work, it is considered the linear form:

$$\Phi\left(\frac{F}{\Psi}\right) = \left(\frac{F}{\Psi}\right) \tag{13}$$

Equation (11) can also be written as follows:

$$\dot{\chi}^p_{\alpha\beta} = \gamma \left\langle \Phi\left(\frac{F}{\Psi}\right)\right\rangle \frac{\partial f}{\partial M_{\alpha\beta}} \tag{14}$$

By multiplying both sides of eq. (14) by $M_{\alpha\beta}$, considering that $f(M_{\alpha\beta})$ is homogeneous of degree one (requirement satisfied by the adopted yield criteria) and applying the Euler theorem, one can obtain:

$$M_{\alpha\beta}\dot{\chi}^p_{\alpha\beta} = \gamma \left\langle \Phi\left(\frac{F}{\Psi}\right)\right\rangle f(M_{\alpha\beta}) \tag{15}$$

From eq. (15) and using eqs. (8) and (10), it can be found that equivalent viscoplastic strain rate is:

$$\dot{\chi}^p_e = \gamma \left\langle \Phi\left(\frac{F}{\Psi}\right)\right\rangle \tag{16}$$

In the present work, the problem is solved using the BEM by considering an initial stress technique in a procedure analogous to that used in Telles and Brebbia [4] and Telles [5] for two- and three-dimensional problems.

Reorganizing the terms of eq. (16), and considering a situation in which $F > 0$, one has:

$$f\left(M_{\alpha\beta}\right) = \Psi(\kappa)\left[1 + \Phi^{-1}\left(\frac{\dot{\chi}^p_e}{\gamma}\right)\right] \tag{17}$$

Equation (17), when compared to the eq. (8), shows the explicit dependence of the yielding surface over the equivalent plastic strain rate.

It can also be demonstrated that the equivalent total strain rate ($\dot{\chi}_e$) is equal to the equivalent elastic strain rate ($\dot{\chi}^e_e$) plus the equivalent plastic strain rate ($\dot{\chi}^p_e$) by considering the following equation

$$\dot{\chi}_e = \dot{\chi}^e_e + \dot{\chi}^p_e \tag{18}$$

where $\dot{\chi}^e_e = \dot{M}_e/D$. Then, one can write

$$\dot{\chi}_e = \frac{\dot{M}_e}{D} + \dot{\chi}^p_e \tag{19}$$

and the substitution of eq. (19) into eq. (17) leads to:

$$f\left(M_{\alpha\beta}\right) = \Psi(k)\left[1 + \Phi^{-1}\left(\frac{\dot{\chi}_e - \dot{M}_e/D}{\gamma}\right)\right] \tag{20}$$

This demonstrates the explicit dependence of $f\left(M_{\alpha\beta}\right)$ over the rate of induced strains/stresses.

Considering eq. (12), the initial moment rates can be evaluated by:

$$\dot{M}^P_{\alpha\beta} = \gamma\langle\Phi\rangle d_{\alpha\beta} \tag{21}$$

in which:

$$d_{\alpha\beta} = C_{\alpha\beta\gamma\theta}\frac{\partial f}{\partial M_{\gamma\theta}} \tag{22}$$

where $C_{\alpha\beta\gamma\theta}$ stands for the components of the fourth order isotropic tensor of elastic constants.

4 Integral equations for nonlinear material Reissner's plates

Integral equations for the generalized displacements at a point ξ, called source point, of the domain Ω can be obtained from a Weighted Residual Method. An equation valid for boundary points can be obtained taking the limits of the integrals of the resulting equation as point ξ tends to the boundary Γ. Karam and Telles [7-8] present the resulting equation applied to elastoplastic problems.

Considering a general nonlinear material form of these integral equations, which allows the application to time dependent problems, in order to perform elastic-viscoplastic analysis, these equations can be written in the following form:

$$c_{ij}(\xi)\dot{u}_j(\xi) = \int_\Gamma u^*_{ij}(\xi,x)\dot{p}_j(x)d\Gamma(x) - \int_\Gamma p^*_{ij}(\xi,x)\dot{u}_j(x)d\Gamma(x) \tag{23}$$

$$+\int_\Omega \left[u^*_{i3}(\xi,x) - \frac{\nu}{(1-\nu)\lambda^2}u^*_{i\alpha,\alpha}(\xi,x)\right]\dot{q}(x)d\Omega(x) + \int_\Omega \chi^*_{\alpha\beta i}(\xi,x)\dot{M}^a_{\alpha\beta}(x)d\Omega(x)$$

Equation (23) holds for internal points with $C_{ij} = \delta_{ij}$, and for boundary points, with $C_{ij} = \delta_{ij}/2$, at smooth boundaries. In that equation, x represents the field point, u^*_{ij}, p^*_{ij} and $\chi^*_{\alpha\beta i}$ are displacement, traction and bending strain tensors related to the fundamental solution to the problem. The second integral is in the Cauchy principal value sense.

Admitting a constant distributed load, $\dot{q}(x) = \dot{q}$, and transforming the domain integral related to this load into a boundary integral, eq. (23) become:

$$c_{ij}(\xi)\dot{u}_j(\xi) = \int_\Gamma u^*_{ij}(\xi,x)\dot{p}_j(x)d\Gamma(x) - \int_\Gamma p^*_{ij}(\xi,x)\dot{u}_j(x)d\Gamma(x) \tag{24}$$

$$+\dot{q}\int_\Gamma \left[v^*_{i,\alpha}(\xi,x) - \frac{\nu}{(1-\nu)\lambda^2}u^*_{i\alpha}(\xi,x)\right]n_\alpha(x)d\Gamma(x) + \int_\Omega \chi^*_{\alpha\beta i}(\xi,x)\dot{M}^a_{\alpha\beta}(x)d\Omega(x)$$

Expressions for u^*_{ij}, p^*_{ij} and $\mathrm{v}^*_{\mathrm{i}}$ can be obtained from Van der Weeën [9] and Karam and Telles [10] and expressions for the tensor $\chi^*_{\alpha\beta i}$ are presented in Karam and Telles [7-8].

5 Integral equations for moments and shear forces at internal points

Moment rates and shear resultant rates at internal points can be evaluated by replacing eq. (24) with $C_{ij} = \delta_{ij}$ and its derivatives with reference to the coordinates of point ξ into eqs. (1) and (2). Substituting the resulting expressions in eqs. (3) and (4) and after some procedures, the following equations (Karam and Telles[8]) arises:

$$\dot{M}_{\alpha\beta} = \int_\Gamma u^*_{\alpha\beta k}\dot{p}_k d\Gamma - \int_\Gamma p^*_{\alpha\beta k}\dot{u}_k d\Gamma + \dot{\mathrm{q}}\int_\Gamma \mathrm{w}^*_{\alpha\beta}\mathrm{d}\Gamma + \int_\Omega \chi^*_{\alpha\beta\gamma\theta}\dot{\mathrm{M}}^{\mathrm{a}}_{\gamma\theta}\mathrm{d}\Omega \tag{25}$$

$$+\frac{\nu\dot{q}}{(1-\nu)\lambda^2}\delta_{\alpha\beta} - \frac{1}{8}\left[2(1+\nu)\dot{M}^a_{\alpha\beta} + (1-3\nu)\delta_{\alpha\beta}\dot{M}^a_{\theta\theta}\right]$$

$$\dot{Q}_\beta = \int_\Gamma u^*_{3\beta k}\dot{p}_k d\Gamma - \int_\Gamma p^*_{3\beta k}\dot{u}_k d\Gamma + \dot{q}\int_\Gamma w^*_{3\beta}d\Gamma + \int_\Omega \chi^*_{3\beta\gamma\theta}\dot{M}^a_{\gamma\theta}d\Omega \tag{26}$$

The last integral in eq. (25) and eq. (26) should be calculated in the Cauchy principal value sense. Expressions for the tensors $u^*_{i\beta k}$, $p^*_{i\beta k}$ and $w^*_{i\beta}$ can be seen in Karam and Telles [10] and expressions for $\chi^*_{i\beta\gamma\theta}$ are presented in Karam and Telles [7-8].

6 Numerical implementation

The numerical implementation was performing in this work by employing continuous and discontinuous quadratic boundary elements and constant triangular internal cells, both with linear geometry.

The following system of equations can be written by applying eq. (24), in discretized form, to all nodal points:

$$\boldsymbol{H}\dot{\boldsymbol{U}} = \boldsymbol{G}\dot{\boldsymbol{P}} + \dot{\boldsymbol{B}} + \boldsymbol{T}\dot{\boldsymbol{M}}^a \tag{27}$$

where $\dot{\boldsymbol{U}}$ is the nodal displacement rate vector, $\dot{\boldsymbol{P}}$ is the nodal traction rate vector, $\dot{\boldsymbol{B}}$ is the vector that contains the influence of the distributed load rate, $\dot{\boldsymbol{M}}^{\boldsymbol{a}}$ is the vector containing the plastic moment rates at the cell points, $\boldsymbol{H}$ and $\boldsymbol{G}$ are square matrices generated from the boundary integrals and $\boldsymbol{T}$ is the matrix formed from the internal cell integrals.

Applying eqs. (25) and (26) in discretized form to all internal cell nodal points, leads to the equations for moment and shear force rates of the form, respectively:

$$\dot{\boldsymbol{M}} = \boldsymbol{G}'\dot{\boldsymbol{P}} - \boldsymbol{H}'\dot{\boldsymbol{U}} + \left(\dot{\boldsymbol{W}}' + \dot{\boldsymbol{V}}'\right) + (\boldsymbol{T}' + \boldsymbol{E}')\dot{\boldsymbol{M}}^a \tag{28}$$

$$\dot{\boldsymbol{Q}} = \boldsymbol{G}''\dot{\boldsymbol{P}} - \boldsymbol{H}''\dot{\boldsymbol{U}} + \dot{\boldsymbol{W}}'' + \boldsymbol{T}''\dot{\boldsymbol{M}}^a \tag{29}$$

where matrices $\boldsymbol{G}'$, $\boldsymbol{G}''$, $\boldsymbol{H}'$ and $\boldsymbol{H}''$ contain the boundary integrals related to the fundamental solution, vectors $\dot{\boldsymbol{W}}'$ and $\dot{\boldsymbol{W}}''$ contain the influence of the distributed load rate, $\dot{\boldsymbol{V}}'$ contains the free term related to the transverse load, $\boldsymbol{T}'$ and $\boldsymbol{T}''$ contain the domain integrals that multiply the plastic moments and $\boldsymbol{E}'$ represents the free term related to the plastic moments.

Considering the boundary conditions, in terms of displacements and tractions, in eq. (27), and reording it, one can obtain:

$$\boldsymbol{A}\dot{\boldsymbol{y}} = \dot{\boldsymbol{f}} + \boldsymbol{T}\dot{\boldsymbol{M}}^a \tag{30}$$

The pre-multiplication of eq. (30) by $\boldsymbol{A}^{-1}$ leads to:

$$\dot{\boldsymbol{y}} = \boldsymbol{R}\dot{\boldsymbol{M}}^a + \dot{\boldsymbol{m}} \tag{31}$$

Then, considering the boundary conditions also in eqs. (28) and (29) and substituting eq. (31) in the resulting equations, one obtains, respectively:

$$\dot{\boldsymbol{M}} = \boldsymbol{S}'\dot{\boldsymbol{M}}^a + \dot{\boldsymbol{n}}' \tag{32}$$

$$\dot{\boldsymbol{Q}} = \boldsymbol{S}''\dot{\boldsymbol{M}}^a + \dot{\boldsymbol{n}}'' \tag{33}$$

In these equations, $\dot{\mathbf{y}}$ is the vector of unknowns, $\dot{\boldsymbol{f}}$, $\dot{\boldsymbol{f}}'$ and $\dot{\boldsymbol{f}}''$ are vectors that contain the prescribed values, including the influence of the transverse load, $\boldsymbol{R} = \boldsymbol{A}^{-1}\boldsymbol{T}$, $\boldsymbol{S}' = \overline{\boldsymbol{T}} - \boldsymbol{A}'\boldsymbol{R}$ and $\boldsymbol{S}'' = \boldsymbol{T}'' - \boldsymbol{A}''\boldsymbol{R}$, with $\overline{\mathbf{T}} = \mathbf{T}' + \mathbf{E}'$. In addition, the vectors $\dot{\boldsymbol{m}}, \dot{\boldsymbol{n}}'$ and $\dot{\boldsymbol{n}}''$ represent the elastic solution to the problem and are expressed, respectively, by $\dot{\boldsymbol{m}} = \boldsymbol{A}^{-1}\dot{\boldsymbol{f}}, \dot{\boldsymbol{n}}' = \dot{\boldsymbol{f}}' - \boldsymbol{A}'\dot{\boldsymbol{m}}$ and $\dot{\boldsymbol{n}}'' = \dot{\boldsymbol{f}}'' - \boldsymbol{A}''\dot{\boldsymbol{m}}$.

7 Solution technique for the elastic-viscoplastic problem

In order to start the incremental process, the maximum equivalent moment calculated at the cell points, M^e_{max}, is reduced to the initial yield moment M_0, and so, the following initial the load factor is set:

$$\lambda_0 = \frac{M_0}{M^e_{max}} \tag{34}$$

If $\lambda_0 > 1$, there will be no viscoplastic analysis and the process will be terminated. Otherwise, if $\lambda_0 \leq 1$, the viscoplastic analysis will be performed for a specific load value. Thus, it will be considered only one step load, or through an incremental load process considering the initial load factor λ_0. Subsequent values of the load factor for the incremental process are calculated by the following recursive equation:

$$\lambda_i = \lambda_{i-1} + \Delta\lambda_i \tag{35}$$

where $\Delta\lambda_i = \beta_i\lambda_0$ is the increment, defined as a given percentage β_i in terms of the load at the first yield.

A simple Euler one-step process has been adopted. A load factor $\lambda_i(t)$ is considered to be a known function of time and, consequently, eqs. (31), (32) and (33) can be integrated on time and become

$$\boldsymbol{y} = \boldsymbol{R}\boldsymbol{M}^p + \lambda_i(t)\boldsymbol{m} \tag{36}$$

$$\boldsymbol{M} = \boldsymbol{S}'\boldsymbol{M}^p + \lambda_i(t)\boldsymbol{n} \tag{37}$$

$$\boldsymbol{Q} = \boldsymbol{S}''\boldsymbol{M}^p + \lambda_i(t)\boldsymbol{n}'' \tag{38}$$

where, vectors $\boldsymbol{m}$, $\boldsymbol{n}$' and $\boldsymbol{n}$'' are the elastic solution at time $t = 0$. Equation (37) is applied after each discrete time step $\left(\Delta\ t = {}^{k+1}t - {}^{k}t\right)$ with the value of initial moments evaluated in the internal cell points, located at the geometric center of the cells, by:

$${}^{k+1}M^p_{\alpha\beta} = {}^{k}M^p_{\alpha\beta} + \Delta\ t\gamma\left\langle {}^{k}\Phi\right\rangle^k d_{\alpha\beta} \tag{39}$$

The load factor $\lambda_i(t)$ is left constant until the steady state condition is reached at the end of each time step. The convergence is monitored in each time step and the values $\Delta\ t\gamma\dot{\chi}^p_e$ or ${}^{k+1}M_e = {}^{k}M_e$ become less than tolerance. So, when the steady state conditions are reached, the process is stopped, or a new load factor is applied.

The proper choice of the size of time increments is of great importance for the success of this process and are determined as presented in Borges and Karam [11].

8 Application

Consider a simply supported circular plate with radius $a = 10\ in$ (254 mm) and thickness $h = 1\ in$ (25.4 mm), subjected to a uniformly distributed load, with the properties: $\upsilon = 0.24$, $E = 10{,}000\ ksi$ (6.867x10^4 MPa) and $\sigma_y = 16\ ks.$ (109.872 MPa). The plate is ideally viscoplastic ($H' = 0$), the fluidity parameter is $\gamma = 0.001$ and the parameters for choosing the time increment are $\eta = 0.08$ and $\eta_0 = 1.5$. The linear form for the function $\Phi(F/\Psi)$ is used and von Mises' yield criterion is adopted.

Due to the symmetry of the problem, only a quarter of the circular plate was discretized. Two meshes were used and are the same to that employed by Karam and Telles [8] for the analysis with classical theory of plasticity. The first mesh has 20 boundary elements and 50 internal cells and the second one has 36 boundary elements and 162 internal cells.

This plate was also analyzed by Armen Jr. et al. [12], using the FEM. Hopkins and Wang [13] performed a limit analysis for the same problem. Armen Jr. et al. [12] obtained collapse load of $\rho = 6.50$ and Hopkins and Wang [13] found limit load $\rho = 6.51$. Karam and Telles [8] found values that converge to $\rho = 6.62$ for mesh 1 while in the present work the obtained value was $\rho = 6.52$. In the present work, it was found $\rho = 6.39$ for mesh 2 while Karam and Telles [8] presented values converging to $\rho = 6.49$.

Fig. 1 shows the load-deflection curves for the central point of the plate for an incremental load process and only one step load, considering $\rho = 6.52$. The results found in the present work are in good accordance with the results presented in Refs. [8,12,13].

Fig.2 presents the plate deflection profile for time $t = 0$ (elastic solution) and time $t = \infty$ when convergence is reached, for the constant load of $\rho = 6.52$, and also the profile found with the use of an incremental loading process with $\beta = 1\%$. It is possible to verify that the displacements tend to be slightly larger when the load is applied in only one step.

Fig. 3 shows redistribution of effective moments with time, considering the load ρ = 6.52 applied in only one step and the parameter $\eta = 0.03$. While the point called *p*, located in the center of gravity of the corresponding cell, is already yielded, the point called *e*, also in the center of gravity of the respective cell, remains elastic.

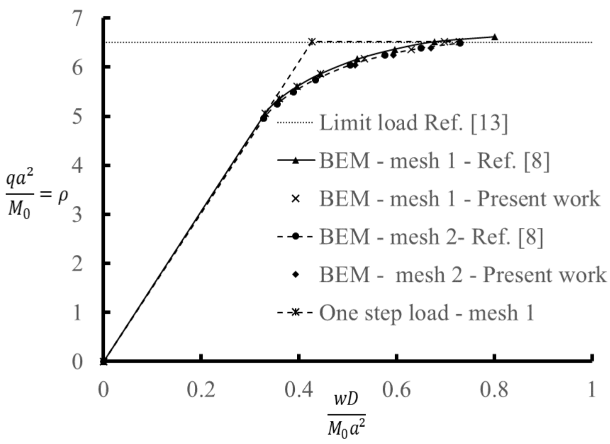

Figure 1. Load-deflection curves for the circular plate.

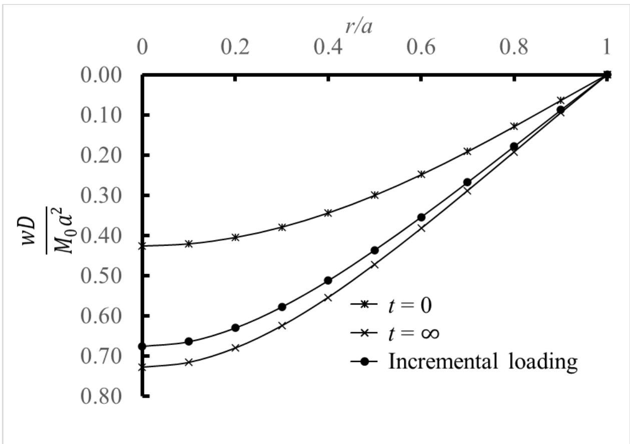

Figure 2. Deflection profiles for one step load $\rho = 6{,}52$, for $t = 0$ and $t = \infty$, and for incremental load process.

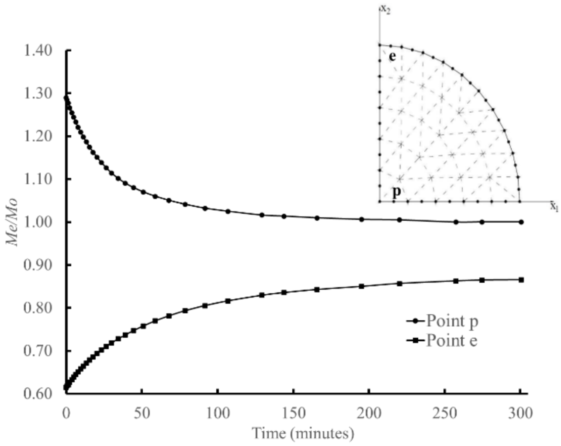

Figure 3. Redistribution of M_e with time for a viscoplastic point (Point p) and elastic point (Point e)

9 Conclusions

In this work, a formulation was presented for elastic-viscoplastic analysis of Reissner's plates, using BEM with an initial stress process. It was considered the constitutive equations due to Perzyna and the yield criteria of von Mises and Tresca. It was developed a computer program for the presented formulation and the discretization of the plates was performed using quadratic boundary elements and constant triangular internal cells. The formulation presents an alternative for elastoplastic solution if the load is applied in small increments and the stationary conditions are reached for each load increment. An application was presented and the results show consistency and good accordance when compared with FEM, BEM and analytical solutions.

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