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A position-based Space-Time formulation for geometrically nonlinear problems

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Abstract. Space-Time finite element methods has been developed over years for solving a series of time-dependent problems like elastodynamics, fluid-structure interaction, fluid flows, advection-diffusion equations and heat transfer problems. The core of this approach is the treatment of time as a dimension of the finite element problem, leading to space-time finite element discretizations. Single-field or two-fields formulation are possible, where the first one uses only displacement as unknowns, while the second uses both displacements and velocities as variables. Some challenges that appear in the Space-Time FEM are the increased size of the equation systems as the precision in time is increased and the 4D meshes representation. Nevertheless, this approach can lead to higher order accuracy in time and direct dynamic spatial re-meshing. On the other hand, time-marching methods are well-known numerical time integrators that have been applied to discrete systems of differential equations obtained from different spatial discretization techniques, including FEM. Most of them deal with approximations for displacements and velocities, and the discrete system of differential equations are solved at each discrete time level taking into account the variable fields from the last time step and the current boundary conditions. Moreover, they can be formulated to present unconditional stability, to present controlled dissipative properties and different orders of accuracy. As a disadvantage, dynamic re-meshing procedures are not directly feasible, as it demands the projection of past time step fields over the new mesh, including projection errors. This work presents a position-based Space-Time FEM formulation for two-dimensional solids with large displacements, using a total Lagrangian description. This formulation is naturally isoparametric and designed directly over the large displacement assumption making the geometric non-linearities intrinsically considered. In order to verify the potential of the formulation, a comparative analysis with the time-marching method *alpha*-generalized is carried out.

Keywords: Space-Time Finite Element Method, Geometric nonlinearity, Time-marching methods

1 Introduction

The Finite Element Method is widely employed for structural mechanics with large displacements, with several important works. Among the important contributions in this field, we can mention Hughes and Carnoy [1], related to shell elements, Schulz and Filippou [2] related to beam elements in a Lagrangian formulation and Crisfield [3] related to solid elements. Motivated by the work of Bonet et al. [4], Coda [5] introduces the positional formulation of the FEM, an alternative approach built over the total Lagrangian description and considering the current positions as unknowns. This approach has been successfully applied to several static and dynamic problems as one can see from Carrazedo and Coda [6], Coda and Paccola [7], including structural contact cases [8, 9]. Regarding to solid elements, it produces a more compact equationing than the displacement-based formulation, and regarding to bars, shells and plates, it avoids rotations as degrees of freedom. Furthermore, the position-based formulation is naturally and truly isoparametric, since its nodal parameters (unknowns) are the current coordinates of the solid.

Time-marching methods are well-known time integrators applied to time-dependent spatially discrete systems of differential equations, which can be obtained with the application of FEM techniques over a continuum domain. These methods provide approximations in time for displacements, velocities and acceleration. If the approximations are based on past and current discrete instants the method is explicit and its stability depends on the adopted time increment (conditionally stable). If the approximation is based on past and future instants, the method is implicit, and therefore, unconditional stability is possible.

Problems involving impact [9] are a challenge for some very well known second order accurate implicit time integrators, such as the Newmark method [10] due to strong nonlinearity and high frequencies. In this sense, studies have been carried about introducing dissipation over high frequencies preserving the accuracy, such as the method proposed by Hu [11], that consists in modifying the Newmark optimal parameters, introducing dissipation in time. Despite being feasible in several cases, as can be seen in Carvalho [8], changing the parameters of the Newmark method requires very small time steps due to the sensitivity of the numerical damping introduced, which can generate a phase error and decrease the accuracy. Another possibility is the *alpha*-Generalized [12] method that can dissipate high frequencies with a better control over damping in time.

Differently from time-marching procedures, the *Space-time* (ST) formulations, introduced by Hughes and Hulbert [13], rellies in the application of the finite element technique over the space-time domain, being the deformation of space with respect to time taken into account automatically. This approach has been improved and applied to a series of time-dependent problems, like linear and nonlinear elastodynamics [14], structural problems coupled with continuum damage mechanics [15], fluid-structure interaction [16] and fluid flows [17].

Two approaches are possible in ST methods: single-field, where the nodal variables are only displacements, and two-fields where the nodal variables are both, displacements and velocities. Also, the problem can be stated from Discontinuous Galerkin method [18] or from Continuous Galerkin method [14]. The numerical accuracy of ST methods depends only in the choice of the shape functions in time direction, however, increasing shape functions order in time also increases the system size, that is the biggest challenge of such methods in addition to the representation of 4D meshes.

In this work, we present a space-time finite element formulation for geometrically nonlinear dynamic 2D elasticity problems. The applied numerical framework is the Positional Finite Element Method, described briefly in the sections 2 and 3. This formulation is compared to the α-generalized time marching method in section section 4 through an example, in order to verify the potential of the formulation. The results and conclusions are discussed in section 5.

2 Solid mechanics

The state of mechanical equilibrium occurs when the variation in the total mechanical energy functional (Π) is null, which translates the principle of stationary energy. In this work, the functional Π is composed by the sum of the potential energy of the external forces ($\mathbb{P}$), strain energy ($\mathbb{U}$) and kinetic energy ($\mathbb{K}$). Therefore, the equilibrium is expressed in variational form and under a total Lagrangian description as

$$\delta\Pi = \underbrace{-\int_{\Gamma_0} \mathbf{q}\cdot\delta\mathbf{y}\, d\Gamma_0 - \int_{\Omega_0} \mathbf{b}\cdot\delta\mathbf{y}\, d\Omega_0}_{\delta\mathbb{P}} + \underbrace{\int_{\Omega_0} \mathbf{S} : \delta\mathbf{E}\, d\Omega_0}_{\delta\mathbb{U}} + \underbrace{\frac{1}{2}\int_{\Omega_0} \rho_0 \ddot{\mathbf{y}}\cdot\delta\mathbf{y}\, d\Omega_0}_{\delta\mathbb{K}} = 0, \tag{1}$$

where $\mathbf{q}$ and $\mathbf{b}$ denote conservative forces distributed along the initial surface Γ_0 and the initial volume Ω_0, respectively, ρ_0 is the initial density, $\mathbf{y}$ denotes the current position vector, $\mathbf{S}$ is the second Piola-Kirchhoff stress tensor, and $\mathbf{E}$ is the Green-Lagrange strain, defined as

$$\mathbf{E} = \frac{1}{2}(\mathbf{A}^T\cdot\mathbf{A} - \mathbf{I}), \tag{2}$$

with $\mathbf{A}$ denoting the deformation gradient, and $\mathbf{I}$ the identity tensor.

The second Piola-Kirchhoff stress is the energetic conjugate of $\mathbf{E}$, and can be defined as $\mathbf{S} = \partial u_e/\partial\mathbf{E}$, in which u_e is the strain energy density, defined by the constitutive model of the material. In this work, we apply the Saint-Venant-Kirchhoff constitutive model, written as:

$$u_e = \frac{1}{2}\mathbf{E} : \mathfrak{C} : \mathbf{E}, \tag{3}$$

where $\mathfrak{C}$ is a fourth order constitutive elastic tensor, defined by $\mathfrak{C} = \frac{\partial^2 u_e}{\partial\mathbf{E}\otimes\partial\mathbf{E}} = \frac{\partial\mathbf{S}}{\partial\mathbf{E}}$, so that the second Piola-Kirchhoff stress tensor can be written as $\mathbf{S} = \mathfrak{C} : \mathbf{E}$.

3 Position-based space-time finite element solution

The discrete space-time domain can be represented by $Q^h \equiv \Omega_0^h \times I^h$, where Ω_0^h is the initial discrete spatial domain, $I^h = (0, T)$ is the discrete time domain and T the final instant. Following works like [15] and [18], we subdivide the time domain to generate space-time slabs. The n^{th} space-time slab is given by $Q_n^h \equiv \Omega_0^h \times I_n^h$, with boundary $\gamma_n^h \equiv \Gamma_0^h \times I_n^h$, where $I_n^h = [t_n, t_{n+1}]$. In turn, each discrete space-time slab Q_n^h is partitioned into n_{el} space-time elements Q^e with a set of n_{nd}^{st} space-time nodes, so that $Q_n^h = \bigcup_{e=1}^{n_{el}} Q^e$.

The strong form of governing equations for a continuum space-time domain is:

$$
\begin{aligned}
\nabla_{\mathbf{x}} \cdot \mathbf{P}^T + \mathbf{b}_0 &= \rho_0 \ddot{\mathbf{y}} && \text{in } Q \equiv \Omega_0 \times I, && (4)\\
\mathbf{y}(\mathbf{x}, t) &= \bar{\mathbf{y}}(\mathbf{x}, t) && \text{on } \gamma^D \equiv \Gamma_0^D \times I, && (5)\\
\mathbf{P}^T \cdot \mathbf{n}_0 &= \mathbf{t}_0(\mathbf{x}, t) && \text{on } \gamma^N \equiv \Gamma_0^N \times I, && (6)\\
\mathbf{y}(\mathbf{x}, 0) &= \mathbf{y}_0(\mathbf{x}) && \text{in } \mathbf{x} \in \Omega_0, && (7)\\
\dot{\mathbf{y}}(\mathbf{x}, 0) &= \dot{\mathbf{y}}_0(\mathbf{x}) && \text{in } \mathbf{x} \in \Omega_0, && (8)
\end{aligned}
$$

where $\nabla_{\mathbf{x}} \cdot (\bullet)$ denotes the Lagrangian divergence of $(\bullet)$, $\mathbf{P}$ is the first Piola-Kirchhoff stress tensor, $\mathbf{b}_0$ is the body force in the initial spatial configuration, ρ_0 is the initial mass density, $\dot{\mathbf{y}}$ is the velocity field. The dots superscript represents time derivatives. The functions $\bar{\mathbf{y}}(\mathbf{x}, t)$ and $\mathbf{t}_0(\mathbf{x}, t)$ are the prescribed position and traction over Dirichlet (γ^D) and Neumann (γ^N) boundaries, respectively. The vector $\mathbf{n_0}$ is the unit outward normal vector to the boundary Γ_0^N.

The first variation of total mechanical energy can be written by multiplying Eq. (4) by a variation of current position $\delta\mathbf{y}$, integrating over space-time domain and making use of the divergence theorem, resulting:

$$
\delta\Pi(y, t) = \int_{Q_n} \rho_0 \ddot{\mathbf{y}} \cdot \delta\mathbf{y}^h \, dQ + \int_{Q_n} \mathbf{S} : \delta\mathbf{E} \, dQ - \int_{Q_n} \mathbf{b}_0 \cdot \delta\mathbf{y}^h \, dQ - \int_{\gamma_n^N} \mathbf{t}_0 \cdot \delta\mathbf{y}^h \, d\gamma = 0. \tag{9}
$$

We consider the meshes to be structured in time direction to ensure the subdivision into time-slabs, which allows each space-time element to be generated by a Cartesian product of spatial and temporal finite elements space of functions. In 2D case, it is easy to visualize the parametric cubic space formed by (ξ_1, ξ_2, θ), so that the space-time shape functions $\mathbf{N}$ are constructed as follows:

$$
\mathbf{N}(\xi_1, \xi_2, \theta) = \mathbf{\Psi}(\theta) \otimes \boldsymbol{\varphi}(\boldsymbol{\xi}) \tag{10}
$$

where $\otimes$ denotes tensor product, $\boldsymbol{\varphi}(\boldsymbol{\xi})$ are the spatial discretization shape functions, obtained with 6 nodes triangles (quadratic Lagrange polynomials) in this work, and $\mathbf{\Psi}(\theta)$ are the temporal shape functions, obtained in this work with one line element with Hermitian cubic polinomials [19]. This approach allows the following approximations:

$$
\begin{aligned}
\mathbf{y}^h(\boldsymbol{\xi}, \theta) &= \mathbf{N}(\boldsymbol{\xi}, \theta) \cdot \mathbf{R}, && (11)\\
\dot{\mathbf{y}}^h(\boldsymbol{\xi}, \theta) &= \dot{\mathbf{N}}(\boldsymbol{\xi}, t) \cdot \mathbf{R}, && (12)
\end{aligned}
$$

where $\mathbf{R} = \begin{bmatrix} \mathbf{Y} & \mathbf{V} \end{bmatrix}$ is the space-time vector of nodal unknowns with position components Y^β and velocity components V^β for each β space-time node.

From now, the fully discrete form of the problem is given by:

$$
-\int_{Q_n^h} \mathbf{N}^T \cdot \mathbf{b}_0 \, dQ - \int_{(\gamma^N)_n^h} \mathbf{N}^T \cdot \mathbf{t}_0 \, d\gamma + \int_{Q_n^h} \rho_0 \mathbf{N}^T \cdot \ddot{\mathbf{N}} \, dQ \cdot \mathbf{R} + \int_{Q_n^h} \mathbf{S}^h : \frac{\partial \mathbf{E}^h}{\partial \mathbf{R}} \, dQ = 0. \tag{13}
$$

Eq. (13) represents a nonlinear system, that is solved by Newton-Raphson method.

4 Numerical example:

A numerical example is simulated in order verify the potential of proposed STFEM formulation through a comparative analysis with *alpha*-generalized method [12]. These method is a more general time integrator, characterized by performing the integration of time in an intermediate instant '$s+1-\alpha$', where the variables are calculated in terms of previous and current values by the linear interpolation. By taking specific α values, the equilibrium equation is evaluated purely on the current time step, falling back to the traditional Newmark integrator [10], as was done in this work.

The problem consists of a cantilever beam subjected to a suddenly and constant load $F = 5.0$kN at point A, as shown in Fig. 1. The material is modelled with the Saint-Venant-Kirchhoff constitutive model, with the following parameters: $\mathbb{E} = 210.0$MPa; $\nu = 0.0$ and; $\rho_0 = 1691.81$kg/m^3. The spatial mesh tested is composed by 24 T6-elements and the time intervals analized are $\Delta t = 1.0 \times 10^{-5}$ s and $\Delta t = 1.0 \times 10^{-6}$ s, for both methods. Time intervals with order higher than those ones ($\Delta t = 1.0 \times 10^{-4}$ s) were also analyzed. However, the answers diverged when using the Space-time method under these conditions. The results are shown in Figs. 2 and 3.

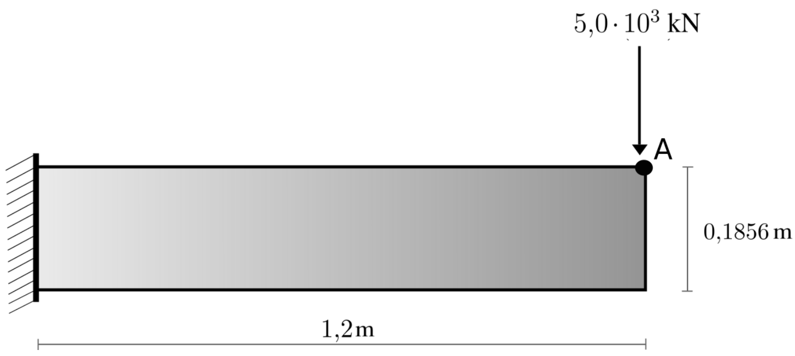

Figure 1. Cantilever beam with suddenly load at free end (point A).

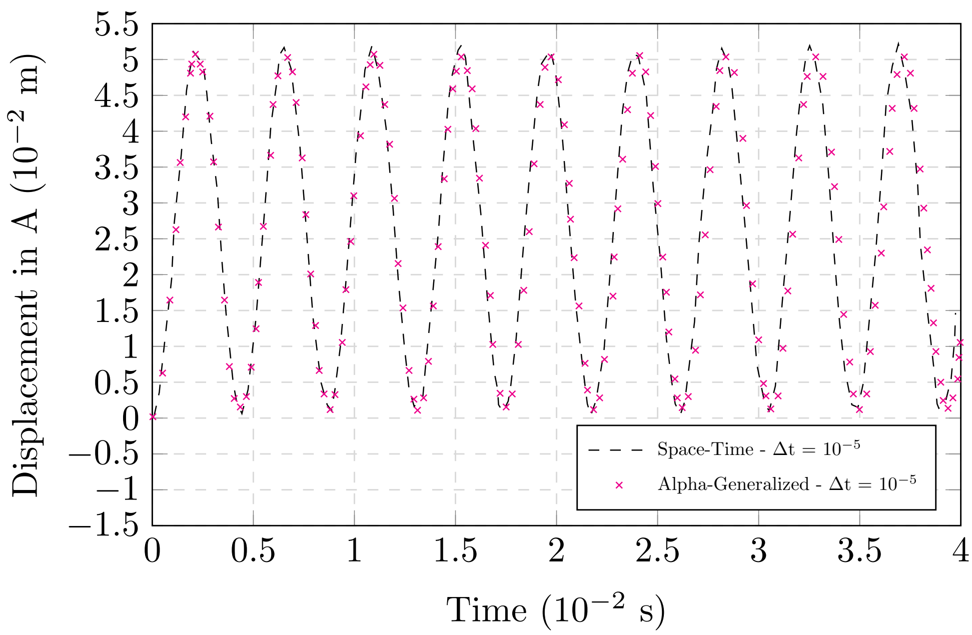

Figure 2. Displacement in a vertical direction at point A versus time for $\Delta t = 10^{-5}$.

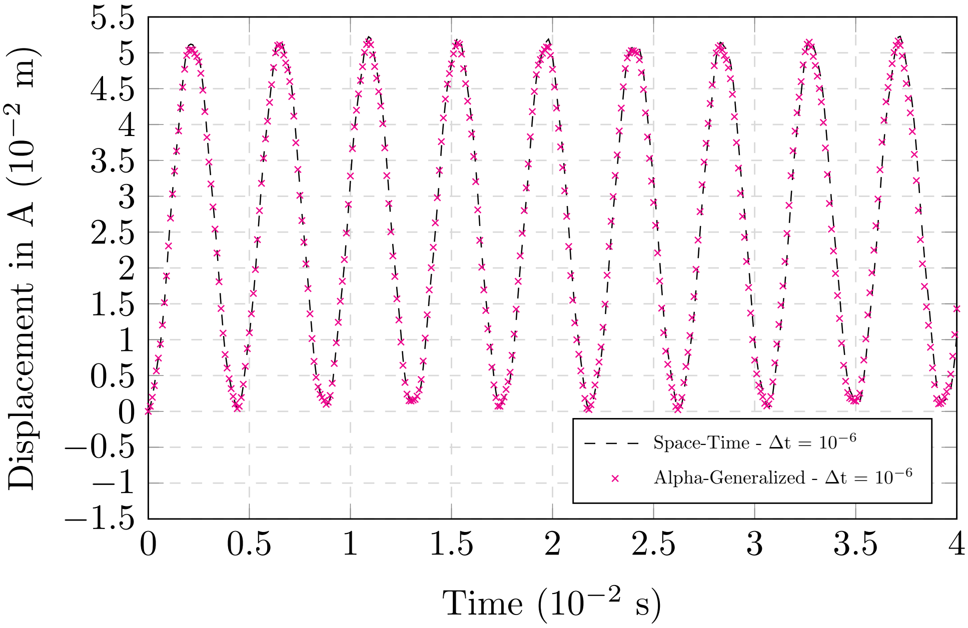

Figure 3. Displacement in a vertical direction at point A versus time for $\Delta t = 10^{-6}$.

5 Conclusion

The present Space-Time Positional FEM formulation is current position-based and, for this reason, is naturally and truly isoparametric. By employing cubic Hermitian polynomials, nodal velocities can be directly related with position approximations. The results from section section 4 compared the ST formulation with time-marching *alpha*-Generalized method in an analysis using finite element method in a geometrically nonlinear dynamic problem. It can be seen that for nonlinear problems such as the beam shown, both methods are equivalent in terms of displacement response for specific time intervals and get closer as the time interval decreases. However, the fact that our Space-time formulation approximates positions with high-order approximation in time (Hermitian functions) makes the problem conditionally stable, requiring small time intervals in the analyses and increasing the processing time (the results diverged with high time intervals). Thus, it is concluded that the ST formulation should gain strength and advantage over the *alpha*-Generalized in problems with high nonlinearity, such as contact/impact problems. In these types of problems, the *alpha*-Generalized integrator is more challenging, due to strong nonlinearity and high frequencies, and the ST formulation can be more stable despite longer processing time. As suggestions for future work, the potential of the ST formulation should be verified for contact/impact problems. Besides, it can be evaluate if worth use other temporal shape functions.

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References

[1] T. J. Hughes and E. Carnoy. Nonlinear finite element shell formulation accounting for large membrane strains. *Computer Methods in Applied Mechanics and Engineering*, vol. 39, n. 1, pp. 69 – 82, 1983.

[2] M. Schulz and F. Filippou. Nonlinear spatial timoshenko beam element with curvature interpolation. *International Journal for Numerical Methods in Engineering*, vol. 50, pp. 761 – 785, 2001.

[3] M. A. Crisfield. *Nonlinear finite element analysis of solids and structures. Volume 1: Essentials*, 1991.

[4] J. Bonet, R. Wood, J. Mahaney, and P. Heywood. Finite element analysis of air supported membrane structures. *Computer Methods in Applied Mechanics and Engineering*, vol. 190, n. 5, pp. 579 – 595, 2000.
[5] H. B. Coda. An exact fem geometric non-linear analysis of frames based on position description. In *International Congress of Mechanical Engineering*. ABCM, 2003.
[6] R. Carrazedo and H. Coda. Alternative positional fem applied to thermomechanical impact of truss structures. *Finite Elements in Analysis and Design - FINITE ELEM ANAL DESIGN*, vol. 46, pp. 1008–1016, 2010.
[7] H. B. Coda and R. R. Paccola. An alternative positional fem formulation for geometrically non-linear analysis of shells: Curved triangular isoparametric elements. *Computational Mechanics*, vol. 40, n. 1, pp. 185–200, 2007.
[8] P. R. P. Carvalho. Análise numérica bidimensional de sólidos com comportamento visco-elasto-plástico em grandes deformações e situações de contato. Mestrado, Departamento de Engenharia de Estruturas, Escola de Engenharia de São Carlos, Universidade de São Paulo, São Carlos, 2019.
[9] D. H. F. R. Moreira. Análise numérica bidimensional de interação fluido-estrutura com contato estrutural. Mestrado, Departamento de Engenharia de Estruturas, Escola de Engenharia de São Carlos, Universidade de São Paulo, São Carlos, 2021.
[10] N. M. Newmark. A method of computation for structural dynamics. *Journal of the Engineering Mechanics Division*, vol. 44, pp. 67–94, 1959.
[11] N. Hu. A solution method for dynamic contact problems. *Computers & Structures*, vol. 63, n. 6, pp. 1053 – 1063, 1997.
[12] J. Chung and G. Hulbert. A time integration algorithm for structural dynamics with improved numerical dissipation: The generalized-alpha method. *Journal of Applied Mechanics*, vol. 60, pp. 371–375, 1993.
[13] T. J. Hughes and G. M. Hulbert. Space-time finite element methods for elastodynamics: Formulations and error estimates. *Computer Methods in Applied Mechanics and Engineering*, vol. 66, n. 3, pp. 339–363, 1988a.
[14] S. Dumont, F. Jourdan, and T. Madani. 4d remeshing using a space-time finite element method for elastodynamics problems. *Mathematical and Computational Applications*, vol. 23, pp. 29, 2018.
[15] S. Wada, R. Zhang, S. R. Mannava, V. K. Vasudevan, and D. Qian. Simulation-based prediction of cyclic failure in rubbery materials using nonlinear space-time finite element method coupled with continuum damage mechanics. *Finite Elements in Analysis and Design*, vol. 138, pp. 21–30, 2018.
[16] Y. Bazilevs, K. Takizawa, and T. E. Tezduyar. *Computational Fluid-Structure Interaction: Methods and Applications*. John Wiley & Sons, Chichester, UK, 2013.
[17] von M. Danwitz, P. Antony, F. Key, N. Hosters, and M. Behr. Four-dimensional elastically deformed simplex space-time meshes for domains with time-variant topology. *International Journal for Numerical Methods in Fluids*, vol. 93, n. 12, pp. 3490–3506, 2021.
[18] T. Hughes and G. Hulbert. Space-time finite element methods for elastodynamics: Formulations and error estimates. *Computer Methods in Applied Mechanics and Engineering*, vol. 66, pp. 339–363, 1988b.
[19] T. C. FUNG. Unconditionally stable higher-order accurate hermitian time finite elements. *International Journal for Numerical Methods in Engineering*, vol. 39, n. 20, pp. 3475–3495, 1996.