Dynamic buckling of slender variable section members to self-weight

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Abstract. We present a study of the dynamic buckling of variable section very slender members due to their self-weight. Modern materials and powerful new analysis methods are leading to the design of very slender aerospace structures that may be prone to instability issues. Elastic stability of such structural systems is a problem inside the scope of Non-Linear Dynamics Analysis Methods. An indicator of instability is when the structure's free vibration frequency tends to zero. Two factors affect these frequency results. First the stiffness, composed of elastic stiffness, always positive and non-zero, that diminishes rapidly with height, and the geometric stiffness, negative for compressive forces, whose absolute value grows as the structure gets taller and heavier. Second, the mass, that also grows with the height of the structures, and is always positive. To access this behavior, we present a one-degree-of-freedom mathematical model of a cantilever vertical member via Rayleigh's Method, adopting a cubic polynomial as shape function. Closed form formulas are obtained for elastic and geometric equivalent stiffness, as well as for equivalent mass, dependent on the member length and transverse section variable characteristics. For some adopted numerical geometric and material properties we use an optimization algorithm to maximize the member length. Check of the formulation is possible for the prismatic member case.

Keywords: dynamic stability, self-weight, Rayleigh's Method, optimization.

1 Introduction

In this paper, a mathematical and numerical study of the dynamic buckling of very slender structures due to their self-weight, a classical problem, is revisited. It is an important issue in several applications such as Eolic energy generators and cell phone equipment support towers, industrial chimneys etc. It is also an possible problem in aerospace engineering, as is the case of vertical rocket structures.

Modern materials and powerful computational analysis methods, usually based on the Finite Element Method, are leading to the design of very slender tall structures that are likely to be prone to instability issues.

Elastic stability of such structures is a problem inside the scope of the Non-Linear Dynamics Analysis Methods. The main issue is the variation of the geometric stiffness.

An indicator of instability is when the structure's free vibration frequency approaches null value. Two main factors affect these frequency results. First the stiffness, composed of elastic stiffness, always positive and non-zero, that diminishes rapidly with height, and the geometric stiffness, negative for compressive forces, whose absolute value grows as the structure gets taller and heavier. Second, the mass, always positive, that also grows with the height of the structures.

In this paper, to access this behavior, we first present a simple one-degree-of-freedom mathematical model derived with Rayleigh's Method [1], adopting a cubic polynomial as shape function. Next, comparisons are made with analytical close solutions available in technical literature such as Timoshenko [2] and Wahrhaftig et al. [3-8].

2 Rayleigh's Method

John William Strutt, 3rd Baron of Rayleigh, presented, in "The Theory of Sound" [1], a simplified method to access the dynamic properties of structures. In the case of vertical beams, the transverse horizontal vibration displacement of its x axis, as shown in Fig. 1, is a function of both the spatial x coordinate and time, and may be written as

$$u = u(x,t) = \phi(x)q(t) \tag{1}$$

where $\phi(t)$ is a dimensionless *shape function*, that must satisfy the geometric boundary conditions and assume unitary value where the $q(t)$ generalize coordinate is chosen. In our model, Fig. 1, that is the horizontal displacement the top section of the beam, where a point vertical load P may be applied, usually self-weigh of a point mass M.

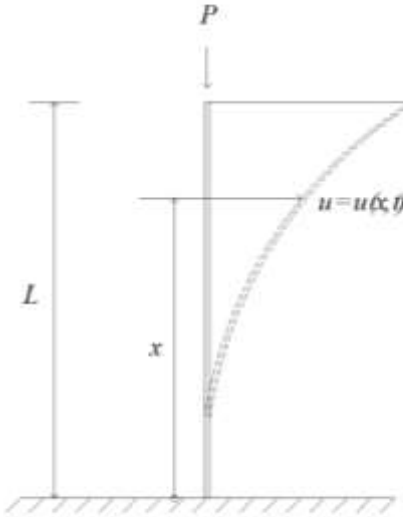

Figure 1. Mathematical model

2.1 Prismatic beam

For model a prismatic, cantilever, homogeneous beam, with modulus of elasticity E, transverse section moment of inertia I, mass per unit length $\overline{m}$, self-weight per unit length $q = \overline{m}g$, an adequate such function is a simple cubic polynomial:

$$\phi(x) = \frac{3x^2}{2L^2} - \frac{x^3}{2L^3} \tag{2}$$

that displays zero displacement and tangent of its x longitudinal axis at the base and null second derivative at the top, related to zero bending moment there. The resulting equivalent elastic stiffness is

$$k_0{}^* = EI\int_0^L (\phi'')^2\,dx = \frac{3EI}{L^3} \tag{3}$$

the equivalent geometric stiffness, due to the point vertical load P (negative if compressive), is

$$k_g^* = P\int_0^L (\phi')^2 dx = \frac{6P}{5L} \tag{4}$$

the equivalent geometric stiffness, due to self-weight, is

$$k_g^{**} = -q\int_0^L (L-x)(\phi')^2 dx = -\frac{3}{8}q \tag{5}$$

and the equivalent lumped mass at the top of the beam is

$$m^* = \overline{m}\int_0^L (\phi)^2 dx = \frac{33\overline{m}L}{140} \tag{6}$$

The first natural frequency, in radians per second, of this mathematical model, is given by

$$\omega_1 = \sqrt{\frac{k}{m^*}} \tag{7}$$

where the total stiffness k is the sum of Eqs. (3), (4) and (5),

$$k = k_0^* + k_g^* + k_g^{**} \tag{8}$$

2.2 Variable section beam

Next, we present a mathematical model of a variable section vertical beam. It intends to represent an annular rocket fuselage with linearly variable diameter and wall thickness. The structure's vertical length is L (m), Young's Modulus E (Pa), material unit mass ρ (kg/m³), average base (larger) diameter d_1 , average top (smaller) diameter d_2 , base (larger) thickness t_1 , top (smaller) thickness t_2 .

Adopting x to be the longitudinal axis, the linear variation of the wail thickness and the average diameter is

$$t = t_1 - \frac{t_1 - t_2}{L} x \tag{9}$$

$$d = d_1 - \frac{d_1 - d_2}{L} x \tag{10}$$

This leads to transverse section area and moment of inertia given by

$$I = \frac{\pi}{8} d^3 t \tag{11}$$

$$A = \pi d t \tag{12}$$

Now we apply Rayleigh's method to compute equivalent stiffness and mass

$$k_0^* = E \int_0^L I(\phi'')^2 dx \tag{13}$$

$$m^* = \rho \int_0^L \rho A \phi^2 dx \tag{14}$$

We use the same Eq. (2) cubic polynomial shape function,

$$\phi = \frac{3x^2}{2L^2} - \frac{x^3}{2L^3} \qquad \phi' = \frac{3x}{L^2} - \frac{3x^2}{2L^3} \qquad \phi'' = \frac{3}{L^3}(L - x) \tag{15}$$

resulting

$$k_0^* = \frac{E\pi}{1120L^3}[(12d_2^3 + 27d_1d_2^2 + 36d_1^2d_2 + 30d_1^3)t_2 + (9d_2^3 + 36d_1d_2^2 + 90d_1^2d_2 + 180d_1^3)t_1] \tag{16}$$

$$m^* = \frac{\rho\pi L}{10080}[(1630\, d_2 + 305\, d_1)t_2 + (305\, d_2 + 136\, d_1)t_1] \tag{17}$$

Let us now turn to the computation of the geometric stiffness of this model due to its own weight, using, as before, Rayleigh's Method and the same Eq. (2) cubic polynomial shape function. Now, in Eq. (5), the beam's own weight q must be a function of x, $q = \rho g\, A$, leading to:

$$k_g^{**} = -\rho g \int_0^L A\,(L - x)(\phi')^2 dx \tag{18}$$

and

$$k_g^{**} = -\frac{\rho g \pi}{1120}[(141d_2 + 87d_1)t_2 + (87d_2 + 105d_1)t_1] \tag{19}$$

3 Numerical results

We adopt L = 14 m, E = 5.000.000.000 N/m², ρ = 1.800 kg/m³, d_1 = 1.3 m, d_2 = 0.8 m, t_1 = 0.3 m, t_2 = 0.2 m, g = 10 m/s², resulting k_0^* = 9.9232x10^5 N/m, k_g^{**} = - 5403 N/m, m^*= 3,662.6 kg.

If geometric stiffness due to self-weight is neglected, we find the first natural undamped frequency to be 2.6197 Hz. In the other hand, if we consider this effect, we get 2.6125 Hz, little significant change in the frequency value being found.

To observe the effect of self-weight on buckling of a cantilever vertical beam, it is necessary to adopt a slenderer beam, reducing, as much as possible, the moment of inertia of the transverse section. Thus, a new model was used, a cantilever aluminum bar.

Numerical parameters adopted for our simulations are: rectangular transverse section 25.4x3.175mm, aluminium elasticity modulus E = 70 GPa and mass per unit volume 2,700 kg/m³, gravity 10 m/s².

In Table 1 we present variation of the first natural frequency with the L free length of the cantilever beam if no point vertical force P at the top is considered.

One will note that zero frequency, corresponding to buckling due to self-weight, occur for L = 2.5924 m, only 0.7% larger than the value L = 2.5747 m given by the analytical result available in Timoshenko [2].

Table 1. Numerical simulations of Rayleigh's mathematical model (P =0).

Free length (m)	**Frequency (rad/s)**
2.0	3.061
2.1	2.584
2.2	2.145
2.3	1.729
2.4	1.314
2.5	0.856
2.5924	0

4 Optimization

Considering the same transverse section and material properties of the aluminum cantilever bar previously described, an optimization problem is posed to find the maximum length of this structure constrained by its first natural frequency being larger or equal to zero. No restriction is made to the resistance of the material to compressive stresses or to possible excessive slenderness.

We used MATLAB Optimization Toolbox to this effect, namely the *fmincon* function, to find the same 2.5924 m maximum length previously obtained.

5 Conclusions

Mathematical research, using Rayleigh's Method, has been carried out to access the effect of self-weight on the buckling of vertical variable section beams, defined as the situation in which the first natural frequency of the structure tends to zero.

It was found that for normal engineering applications, this effect is not significant. Only for extremely slender bars it is possible to observe this influence.

An optimization problem was also posed to find the maximum length of such a structure and solved via the MATLAB Optimization Toolbox *fmincon* function. Results agree very well with closed form solutions available in the literature, such as Timoshenko [2] and Wahrhaftig et al. [3-8].

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