Analysis of the conditioning of the inertia matrix in acoustic models of the Boundary Element Method with Direct Interpolation

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Abstract. After to be solved problems given by Helmholtz, Diffusion-Advection, and Poisson equations, this work presents the results of the Direct Interpolation Method concerning acoustic wave cases to homogeneous media. The main objective is to establish a link between the stability of the discrete model with the conditioning level of the dynamic matrix, which is generated through a sequential of radial basis functions. Several conditioning norms were verified and applied to several linear boundary element meshes. A classical problem of wave propagation in scalar fields was chosen for the numerical analysis.

Keywords: Boundary element method, Acoustic problems, Direct Interpolation technique.

1 Introduction

The accurate and agile solution of dynamic problems through numerical approximations is still one of the most challenging problems in engineering for all numerical methods. Regarding the Boundary Element Method (BEM), there is a classic formulation, which employs the time-dependent fundamental Green solution, strictly associated with the dynamic problem [1]. However, despite its elegance, excellent results and mathematical consistency, this formulation is complex and demand large computational effort. Concerning more simpler and flexible BEM formulations, although subject to numerical inaccuracies, for many years the main BEM alternative aimed at solving dynamic problems was the Dual Reciprocity formulation (DRBEM) [2] and initially applied to the solution of free vibration problems in structures. Due to its versatility, DRBEM was extended and applied to the Poisson equation, Helmholtz, Transient diffusion, diffusive-advective and acoustic waves, and other models [3].

The DRBEM employs a simpler fundamental solution and uses radial basis functions [4] to interpolate part of the kernel of domain integrals. For such interpolation to be processed with reasonable precision, it is necessary to introduce interpolating points inside the domain. The wave propagation cases, governed by equations of a hyperbolic nature, demand many poles which, in turn, also collaborate in introducing degrees of freedom to the system, which are vital for adequately describing dynamic motion. However, the loss of stability in the response of the DRBEM in the solution over time of transient problems, especially the dynamic cases is a well-known phenomenon but still not completely elucidated [5]. Maybe it is due to the use of time advance schemes or direct integration techniques, used in the Finite Element Method and the Finite Difference Method. The cited domain methods have accuracy and consistency problems for large steps, but also work well for small time increments. The DRBEM, on the other hand, is unstable for very short times, which produce discomfort in the application of the method, due to the risk of obtaining spurious solutions.

The step-by-step integration schemes used by domain methods always induced uncertainties, since the BEM discretization is limited exclusively to the boundary. There is no way to make an adequate comparison between finite difference cells and/or finite elements with boundary elements, which have one dimension less. Thus, the idea of establishing a suitable temporal integration step, expressed by the Courant-Friedrichs-Lewi condition (CFL) [6], in which the space traversed by the wave in a time interval cannot be greater than the length of a boundary element it is only a preliminary device, to overcome a still unresolved problem. However, the CFL criteria cannot be directly applied without caveats in the BEM. However, other factors must be considered: the classic BEM matrices related to stiffness are also not symmetric [7]; the inertia matrix is generated through approximations with radial basis functions. However, DRBEM runs into other very serious numerical problems. First, DRBEM is not a simple interpolation technique; its formulation works with two primitive functions of the original interpolation function that can produce additional harmful numerical effects [8]. Second, it presents certain numerical inaccuracies in cases where many internal poles are necessary, generating matrix ill-conditioning matrix problems. In addition, three-dimensional applications showed a strong restriction on the use of functions that were successful in two-dimensional applications [9]. Although DRBEM applications can be found to solve two-dimensional Helmholtz problems [10], the same does not occur with three-dimensional problems. The absence of DRBEM applications in three-dimensional problems is certainly related to its unsatisfactory performance. Many researchers [11, 12] did many simulations with various types of radial functions in two-dimensional problems, but these tests were not repeated for three-dimensional cases.

More recently, a new technique based on the use of radial functions was proposed, called Direct Interpolation with Boundary Elements (DIBEM) [13]. The DIBEM technique, like DRBEM, also approximates non-self-adjoint nuclei by sequences of radial basis functions. However, despite being similar to DRBEM in many respects, the DIBEM formulation does not require the construction of two auxiliary matrices that multiply the classical boundary elements matrices H and G, since it directly addresses the integral domain, similar to what is done in an interpolation procedure using a primitive radial basis function. Unlike DRBEM the entire kernel of the domain integral is interpolated, including the fundamental solution. The DIBEM procedure was successfully applied to solve Poisson [14], Helmholtz [15] and Advection Diffusion [16] problems. In these applications, the performance of DIBEM was clearly superior to DRBEM. Only the transformation of a domain integral into a boundary integral makes DIBEM different from a typical interpolation procedure. A wider range of different radial functions and a large number of poles or base points can be used without numerical problems.

Regarding to stationary wave problems, DIBEM greater robustness and accuracy were demonstrated by comparison with DRBEM results [17], as both formulations use fundamental solutions independent of frequency. However, in the dynamic response, the instability problem for reduced integration steps persisted, although for significantly smaller values. Therefore, in this work, the results obtained by the DIBEM procedure solving acoustic wave propagation problems are analyzed, seeking to relate the stability of the discrete model with the degree of conditioning of the dynamic matrix. Several conditioning norms were compared, applied to several meshes constituted by linear boundary elements, used in the solution of classical problems of wave propagation in scalar fields. A positivity study was also undertaken, in addition to curves that relate the degree of mesh discretization as the minimum integration step capable of generating a stable solution.

2 Governing Integral Equation

Considering the homogeneous acoustic wave equation, given by the Eq. (1), in which a temporal response $u(\boldsymbol{X})$ is produced in a linear system by a variable excitation [18]:

$$u_{,\mathrm{ii}}(\mathbf{X}) = \left(\frac{1}{\mathrm{c}^2}\right)\ddot{u}(\mathbf{X}). \tag{1}$$

As usual, Dirichlet and Neumann boundary conditions [6] should be applied on the two-dimensional boundary, and initial domain conditions must be known on the domain. Thus, considering the mathematical procedures well-known in the context of BEM [7], the inverse integral form associated with Eq. (1) results in the following equation:

$$c(\boldsymbol{\xi})u(\boldsymbol{\xi}) + \int_{\Gamma} u(\boldsymbol{X})q^*(\boldsymbol{\xi};\boldsymbol{X})d\Gamma - \int_{\Gamma} q(\boldsymbol{X})u^*(\boldsymbol{\xi};\boldsymbol{X})d\Gamma = -\frac{1}{k^2}\int_{\Omega} \ddot{u}(\boldsymbol{X})\, u^*(\boldsymbol{\xi};\boldsymbol{X})d\Omega. \tag{2}$$

In Eq. (2), the term $\boldsymbol{X}$ represents the field point, any point related to the domain $\Omega(\boldsymbol{X})$, limited by the boundary $\Gamma(\boldsymbol{X})$. The base point of the integrations is called the source point $\boldsymbol{\xi}$. The coefficient k means the speed of propagation of the acoustic waves. The term $c(\boldsymbol{\xi})$ is a coefficient related to the location of the source point $\boldsymbol{\xi}$ relative to $\Omega(\boldsymbol{X})$ and its smoothness [7]. It should be noted that $u(\boldsymbol{X})$ is the scalar potential and $q(\boldsymbol{X})$ is its normal derivative; $u^*(\boldsymbol{\xi};\boldsymbol{X})$ is the fundamental solution correlated with Laplace's problem and $q^*(\boldsymbol{\xi};\boldsymbol{X})$ is its normal derivative [7]:

$$u^*(\boldsymbol{\xi};\boldsymbol{X}) = -\frac{ln[r(\boldsymbol{\xi};\boldsymbol{X})]}{2\pi}, \tag{3}$$

$$q^*(\boldsymbol{\xi};\boldsymbol{X}) = -\frac{1}{2\pi r(\boldsymbol{\xi};\boldsymbol{X})} r_i(\boldsymbol{\xi};\boldsymbol{X}) n_i(\boldsymbol{X}). \tag{4}$$

In Eq. (3) and (4), $r(\boldsymbol{\xi};\boldsymbol{X})$ is the Euclidean distance between the source point $\boldsymbol{\xi}$ and any source point $\boldsymbol{X}$; and $n_i(\boldsymbol{X})$ is the external normal to the contour at point $\boldsymbol{X}$.

3 Direct Interpolation Technique

Using DIBEM, the approximation given by radial functions includes the fundamental solution, which depends on the source point. For this reason, to avoid singularities, a regularization procedure is used as a strategy [15], according to the following artifice:

$$c(\boldsymbol{\xi})u(\boldsymbol{\xi}) + \int_\Gamma u(\boldsymbol{X})q^*(\boldsymbol{\xi};\boldsymbol{X})d\Gamma - \int_\Gamma q(\boldsymbol{X})u^*(\boldsymbol{\xi};\boldsymbol{X})d\Gamma = \frac{1}{c^2}\left[\ddot{u}(\boldsymbol{\xi})\int_\Omega u^*(\boldsymbol{\xi};\boldsymbol{X})\,d\Omega - \int_\Omega \ddot{u}(\boldsymbol{X})\,u^*(\boldsymbol{\xi};\boldsymbol{X})d\Omega\right]$$
$$-\frac{1}{c^2}\left[\ddot{u}(\boldsymbol{\xi})\int_\Omega u^*(\boldsymbol{\xi};\boldsymbol{X})\,d\Omega\right]. \tag{5}$$

In Equation (5), an additional domain integral is created, which is easily transformed into a boundary integral, through the Galerkin Tensor G* [7]:

$$\ddot{u}(\boldsymbol{\xi})\int_\Omega u^*(\boldsymbol{\xi};\boldsymbol{X})\,d\Omega = \ddot{u}(\boldsymbol{\xi})\int_\Omega G^*_{,ii}(\boldsymbol{\xi};\boldsymbol{X}) = \ddot{u}(\boldsymbol{\xi})\int_\Gamma G^*_{,i}(\boldsymbol{\xi};\boldsymbol{X})n_i\,d\Gamma\,. \tag{6}$$

The entire kernel of the domain integral is interpolated using DIBEM, as shown below:

$$[\,\ddot{u}(\boldsymbol{X}) - \ddot{u}(\boldsymbol{\xi})]u^*(\boldsymbol{\xi};\boldsymbol{X}) \cong \ddot{\alpha}^i F^i(\boldsymbol{X^i};\boldsymbol{X}) = \ddot{\alpha}^i \psi^i{}_{,kk}\,(\boldsymbol{X^i};\boldsymbol{X}). \tag{7}$$

The DIBEM also uses an auxiliary primitive function $\psi^i(\boldsymbol{X^i};\boldsymbol{X})$. The same radial basis functions (RBF) used in DRBEM can be employed. In this article the simple radial function is used. Due to space issues, the matrix treatment of this equation will not be discussed, but can be obtained from previous works [15]. Therefore, the final system can be written as follows:

$$[\boldsymbol{H}]\{\boldsymbol{u}\} - [\boldsymbol{G}]\{\boldsymbol{q}\} = [\boldsymbol{M}]\{\ddot{\boldsymbol{u}}\}. \tag{8}$$

In Eq. (8), the matrices **[H]** and **[G]** are classic BEM matrices that appear from the discretization operations involving the integral of the fundamental solution and its normal derivative, respectively. The matrix **[M]** corresponds to the acoustic inertia property, while **{u}** and **{q}** are vectors that contain the values of the potential and its derivative at the nodes.

4 Time Discretization Schemes

Just unconditionally unconditionally stable time marching schemes should be used with the DIBEM. However, there are particularities that must be considered. The stability concept usually defined to domain methods does not apply directly to the BEM, which is a boundary method. Second, the BEM has a mixed formulation, whose the potential and it's normal derivative are calculated simultaneously. In dynamic problems, higher frequencies, usually calculated numerically with low precision, strongly influence potential derivatives. Thus, for temporal discretization, the most suitable is the Houlbolt time advance scheme, which approximates the acceleration in the form:

$$\ddot{\boldsymbol{u}}_{n+1} = \frac{2\boldsymbol{u}_{n+1} - 5\boldsymbol{u}_n + 4\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-2}}{\Delta t^2} \,. \tag{9}$$

Where Δt is the discretization time interval and n are the temporal instants. Substituting in the equation for elastic wave problems, one has:

$$(2\overline{\boldsymbol{M}} + \Delta t^2 \overline{\boldsymbol{H}})\boldsymbol{u}_{n+1} - (\Delta t^2 \overline{\boldsymbol{G}})\boldsymbol{q}_{n+1} = (5\overline{\boldsymbol{M}})\boldsymbol{u}_n - (4\overline{\boldsymbol{M}})\boldsymbol{u}_{n-1} + (\overline{\boldsymbol{M}})\boldsymbol{u}_{n-2}. \tag{10}$$

Houbolt scheme is unconditionally stable, but instability was observed with the reduction of the time interval (Δt) in the computational BEM experiments.

5 Numerical Simulations

The problem of the bar subjected to sudden load is solved, due to its known numerical difficulties, since all odd vibrational modes contribute with dynamic response. The potentials correspond to the displacements and its results at the right end of the bar are displayed. The geometric features and boundary conditions are shown in Fig.1:

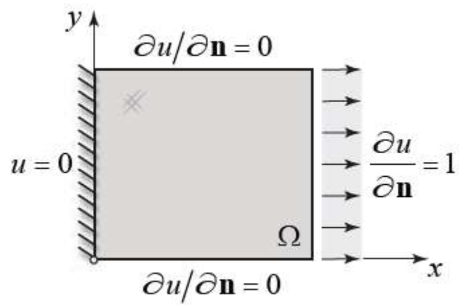

Figure 1. Bar remained the Impact Load and Boundary Conditions.

Regular meshes with linear boundary elements (BE) and double nodes in the corners were used. The experiments start with the simplest mesh with 80 BE and 16 poles, aiming to identify the minimum integration step, which precedes the instability. It is verified that instability occurs for integration steps smaller than or equal to 0.044 seconds, as shown in graph on the left in Fig. 2. Above this value, the response is stable, as can be seen in graph plotted on the right.

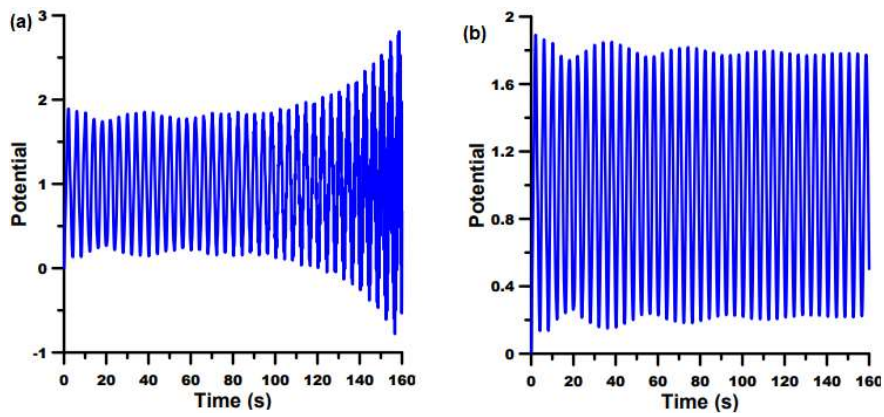

Figure 2: (a) Response with 80BE and 16 poles using Δt=0.044s; (b) Response with Δt=0.045s.

The Houbolt scheme has strong numerical damping, whose intensity increases with the choice of large integration steps. Thus, the larger the step, the more distorted the response is in terms of accuracy, mainly due to the introduction of fictitious damping, as can be seen in Fig. 3, where a step equal to 0.09 seconds was used:

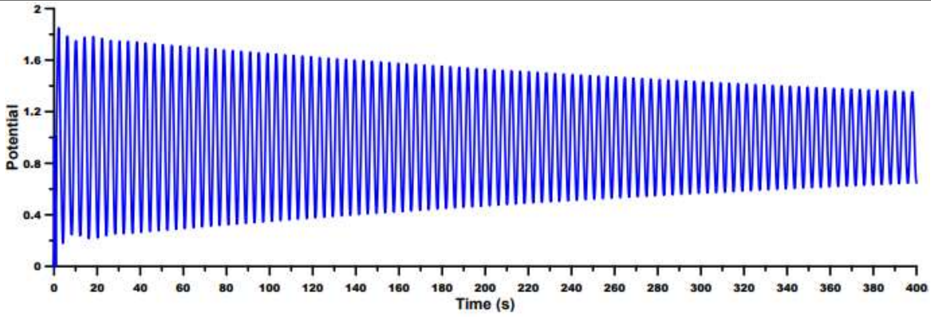

Figure 3: Response with 80BE and 16 poles using Δt=0.09s.

The possibility of employing small steps implies being able to represent high vibrational modes in the response, with small effect of fictitious damping. Fig. 4 shows results achieved using a finer mesh with 640BE and 625 poles.

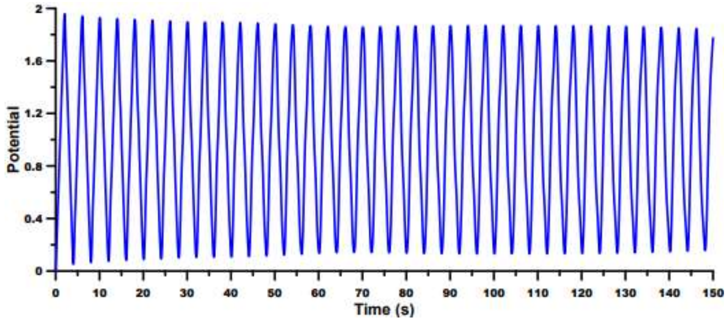

Figure 4: Response with 640BE and 625 poles using Δt=0.015s.

In Figure 5, the relationship between time step and internal points was constructed, taking into account the numerical stability for the problem in question, considering the boundary element meshes and the minimum integration step (in seconds) for stable integration of the answer.

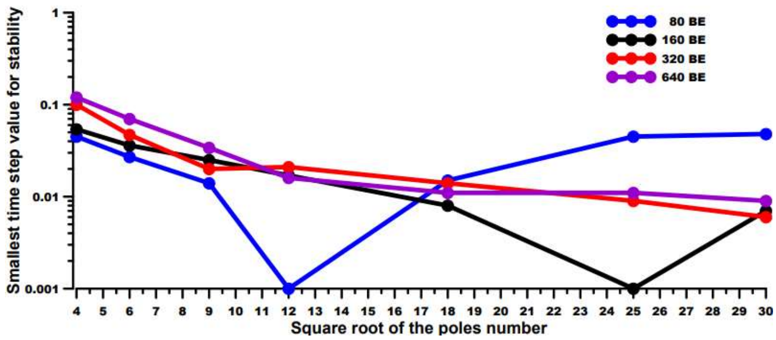

Figure 5: Smallest time step for stability of response as a function of number of poles.

In Figure 5 it is possible to see that there is an optimal configuration related to the combination between the refinement of the contour mesh and the number of internal points, which results in excellent results. It is not yet possible to establish the parameters to find such an optimal configuration. It is also inferred that it is possible to have mesh configurations with the lowest performance.

6 Numerical Matrix analysis

Matrix conditioning and stability of algorithms are aspects that can be linked. An introductory study of conditioning of the DIBEM inertia matrix is now performed. In matrices with a large conditioning number small changes to the input produces large changes in output. There are many norms to evaluate the conditioning number.

Four norms were compared considering the degree of mesh refinement; but as expected, the curves are strongly similar. Thus, just the results obtained for Euclidian norm are shown in Fig. 6. As observed, the coarse boundary meshes have a small conditioning number; however, its numerical results are inaccurate, as shown previously. This implies that conditioning number is not the factor capable of explaining the loss of the stability of the DIBEM marching scheme for small time steps. Moreover, the conditioning number of the finer meshes lies relatively uniform for large number of poles.

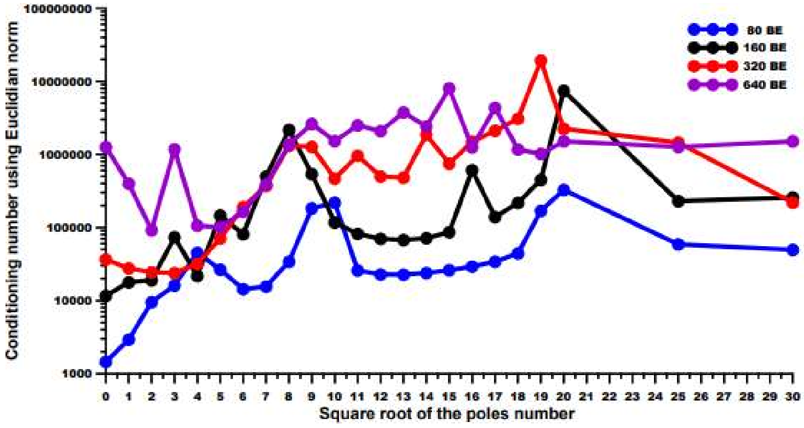

Figure 6: Conditioning number according Euclidian norm as a function of number of poles.

The positivity of a matrix is another important factor in analyzing aspects concerning the consistency of discrete numerical methods. However, it is usually placed in the background because most discrete numerical methods work with symmetric matrices, generating real eigenvalues, but this is not the case for the boundary element method. Thus, complex values are observed in eigenvalues simulations using DRBEM and DIBEM eigenvalues associated with high frequencies. Really, all discrete methods do not have the capability to calculate high frequencies; the low frequencies are always calculated accurately, and usually, they are more important. Non-symmetric BEM matrices produce complex eigenvalues due to the loss of precision of the method in the representation of the higher vibrational modes. Anyway, an eigenvalue analysis of the DIBEM inertia matrix relative to the degree of refinement can be given an idea of positivity. There are also some different norms for measuring positivity. Fig. 7 show the number of complex eigenvalues in DIBEM matrices relatively to the number of poles. It can be realized that the finer boundary meshes without poles have a more significant proportion of real eigenvalues. Thus, the introduction of poles implies a meaningful increase of complex eigenvalues, which seems to stabilize around 35%. It suggests that the high amount of non-real eigenvalues may be an essential factor for the loss of stability.

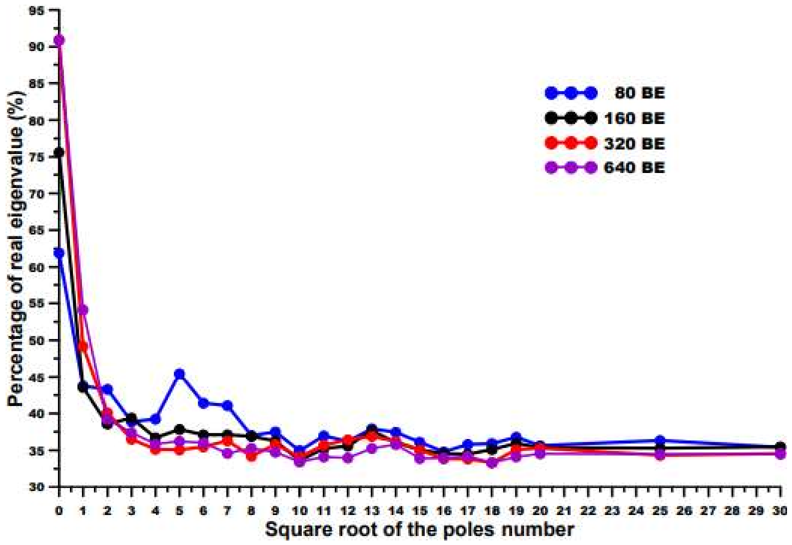

Figure 7: Percentage of real eigenvalues according norm for positiveness as a function of number of poles.

7 Conclusions

Observing the results presented here and comparing them with the content in the literature, it can be preliminarily concluded that the DIBEM formulation offered more accurate results in a larger range of integration intervals than the DRBEM formulation, in agreement with what was observed in previous tests. However, a criterion for a priori identification of a minimum step from which numerical stability is produced still needs to be better studied. The evaluation of the conditioning number of the DIBEM matrices did not conclusively reveal the factor responsible for the loss of stability. Despite the number of poles increasing the ill-conditioning of the inertia matrix, on the other hand, there is a meaningful gain in the quality of results. Regarding the issue of matrix positivity, it is observed that the introduction of numerous poles implies a significant amount of complex eigenvalues, which can disturb the dynamic response for small time steps. In this sense, new problems need to be solved, in order to compose a more precise reference on the value of this minimum step and also on other numerical particularities of the DIBEM in dealing with acoustic problems, such as the radio between the density of the poles on the boundary mesh and internally, and also the elimination of complex eigenvalues in time response.

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