Sensitivity analysis of flexible multibody systems with nonlinear beams

J.C. García Orden[1], J.J. Arribas Montejo[1], F. Gabaldón Castillo[1], D. Dopico Dopico[2]

[1]*Dept. de Mecánica de Medios Continuos y Teoría de Estructuras. ETSI Caminos, Universidad Politécnica de Madrid (UPM)*
C/ Prof. Aranguren 3, 28040, Madrid, Spain
juancarlos.garcia@upm.com
[2]*Dept. de Ingeniería Naval e Industrial. Universidad de A Coruña (UDC)*
C/ Mendizábal s/n, 15403, Ferrol, Spain
daniel.dopico@udc.es

Abstract. Optimization of the dynamics of multibody systems is an active area of research with many important applications in different fields. Among many available optimization techniques, gradient methods are very versatile and popular; and one of its main ingredients is the computation of sensitivities. Sensitivities provides information about how the coordinates of the system change with time when the parameters change. Since multibody systems are typically represented by systems of nonlinear differential equations (or algebraic-differential systems), sensitivities are computed evaluating the corresponding derivatives respect the parameters around the reference movement. These derivatives (sensitivities) depends on time and are the solutions of a system of linear differential equations (with variable coefficients). Their computation may be performed after the solution for the dynamics, or simultaneously with it. Sensitivity analysis of mechanisms exclusively composed by rigid bodies has been studied in many works of the literature. However, analysis dealing with flexible mechanisms are rarer. In this work we show the results of a sensitivity analysis of special systems, where the flexible parts are slender beams represented by a nonlinear beam model. Their sensitivity contributions are computed analytically improving the efficiency and accuracy of the computations. What is more, a robust and physically intuitive approach based on a finite-difference method is presented for obtaining preliminary sensitivity results, that provide a valuable tool for developing and validate the previously described analytical approach. Some simple numerical examples are presented showing the performance of the proposed approach.

Keywords: Multibody, Flexible, Sensitivity, Beam

1 Introduction

The sensitivity analysis of the dynamics of mechanical multibody systems is a valuable tool for improving their design an optimization. This type of analysis is quite common for systems composed by rigid bodies, but rarer for those incorporating flexible parts. For these flexible models, the traditional approach is to use finite-difference schemes, due to the complexity of the formulation, compromising their computational cost. In this paper we explore the possibilities of employing an analytical approach, taking advantage of the simplicity of the formulation of the recently proposed beam model [1]. The following sections provide some details of this beam model, how the sensitivity problem is stated and solved, and a first assessment about the performance of the analytical approach in a simple example.

2 Beam model

As explained in detail in [1], the model for the beam is a collection of n identical deformable truss members (for brevity, we will refer to them just as trusses or segments, identifying them with straight elements that only withstand tensile or compressive forces) with regular section moving in a three-dimensional Euclidean space. Each

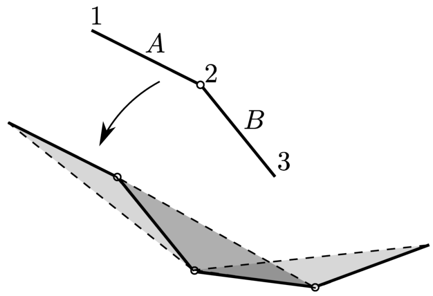

Figure 1. *Overlapped sets of trusses*

truss is defined by two nodes, and two consecutive trusses share one node, as illustrated in Figure 1; thus, the total number of nodes is $N = n + 1$.

The motion of the beam at time $t \in [0, T]$ is determined by vectors $\mathbf{q}(t), \mathbf{p}(t) \in \mathbb{R}^{3N}$ that collects the Cartesian inertial coordinates of the position and momentum vectors of the nodes $\mathbf{r}_i, \mathbf{p}_i; i = 1, ..., N$. This special rotation-free parametrization defines a linear finite-dimensional configuration space that greatly determine the simplicity of the formulation, as will be explained next.

The axial response of the beam is represented by the deformation of the trusses, each of them being a 1D element that can experience large movements and deformation in a 3D space. This deformation is a pure stretch and is related with the internal forces through an hyperelastic potential. On the other hand, bending response is represented by the misalignment of consecutive trusses and is governed by another potential. The sum of both potentials is an approximation of the strain energy of the beam.

2.1 Axial response

We will consider a single truss labelled i where we collect the Cartesian coordinates of the two nodes in the 6×1 vector $\mathbf{q}_i = (\mathbf{r}_i^1 \; \mathbf{r}_i^2)^{\mathrm{T}}$. Let us denot the stretch $\lambda_i = r_i / r_0$ and $r_i = ||\mathbf{r}_i|| = ||\mathbf{r}_i^2 - \mathbf{r}_i^1||$. $r_0 = ||\mathbf{r}_0||$ and a_0 are the initial length and sectional area of the truss respectively. It can be shown that the force vector for a 1D hyperelastic material model is:

$$\mathbf{F}_i = a_0 D_{\lambda_i} W_i \left\{ \begin{array}{c} \mathbf{e}_i \\ -\mathbf{e}_i \end{array} \right\}, \text{ with the notation } D_{\square}(\cdot) = \frac{\partial(\cdot)}{\partial \square} \tag{1}$$

$W_i(E, \lambda_i)$ being the hyperelastic potential and $\mathbf{e}_i = \mathbf{r}_i / r_i$. Two example of such potentials are the logarithmic (non-linear) and linear, adequate for large and small strains respectively:

$$W^{nonlin} = \frac{E}{2} (\ln \lambda)^2 \quad , \quad W^{lin} = \frac{E}{2} (\lambda_i - 1)^2 \tag{2}$$

the second resulting from linearizing the first for $\lambda_i \simeq 1$. The total force vector $\mathbf{F}$ results from the assembly of the elemental vectors: $\mathbf{F}^{ax} = \mathop{\mathbf{A}}_{i=1}^{n} \mathbf{F}_i$. Note that if the beam is discretized in n identical segments, $\lambda_i = r_i n / L_0$, being L_0 the total initial length of the beam.

2.2 Bending response

The bending response is represented by means of a calibrated potential, which is an approximation of the bending strain energy of the beam. This potential depends on the beam coordinates and on a single paramater α, which can be interpreted as the penalty parameter associated to a constraint imposing all segments to remain aligned. This parameter is calculated beforehand for a given discretized beam selecting a configuration where the approximation is exact. Details of this calculations can be found in [1]. The point of departure is the same beam composed by n identical trusses where we define the $n-1$ sets that result from grouping two consecutive segments, such that they overlap as shown in Figure 1 with grey shades. The constraint associated with the alignment of the three-node element labelled j is $\phi_j = ||\mathbf{a}_j \times \mathbf{b}_j||$ with auxiliary vectors $\mathbf{a}_j = \mathbf{r}_j^2 - \mathbf{r}_j^1$ and $\mathbf{b}_j = \mathbf{r}_j^3 - \mathbf{r}_j^2$, and the

9×1 internal force vector $\mathbf{F}_j$ is:

$$\mathbf{F}_j = -\alpha\phi_j D_{\mathbf{q}_j}\phi_j^{\mathrm{T}} \quad \text{with} \quad D_{\mathbf{q}_j}\phi_j^{\mathrm{T}} = \left\{ \begin{array}{c} -\mathbf{b}_j \times \mathbf{n}_j \\ (\mathbf{a}_j + \mathbf{b}_j) \times \mathbf{n}_j \\ -\mathbf{a}_j \times \mathbf{n}_j \end{array} \right\} \quad \text{and} \quad \mathbf{n}_j = (\mathbf{a}_j \times \mathbf{b}_j)/\phi_j \tag{3}$$

Again, the total force vector is the assembly of the elementary contributions $\mathbf{F}^{bend} = \mathop{\mathbf{A}}_{j=1}^{n-1} \mathbf{F}_j$. The penalty value α is unique for the whole beam and is computed as $\alpha/\alpha_0 = n^3(n^2+2)$ with $\alpha_0 = EI/L_0^5$, E being the Young modulus of the material, I the sectional inertia and L_0 the total initial length of the beam.

3 Sensitivity analysis

Sensitivity analysis is usually the initial stage of an optimization process of a dynamical system, looking for the extreme of an objective functional ψ, with the general expression:

$$\psi = \int_{t_0}^{t_F} g(\mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\rho}) \, \mathrm{dt} \tag{4}$$

$\boldsymbol{\rho}$ being a vector containing the parameters of the model. This problem is typically solved employing a gradient method, that requires the computation of the gradient of the funcional respect the parameters; aplying the chain rule:

$$D_{\boldsymbol{\rho}}\psi = \int_{t_0}^{t_F} \left[(D_{\mathbf{q}}g)(D_{\boldsymbol{\rho}}\mathbf{q}) + (D_{\dot{\mathbf{q}}}g)(D_{\boldsymbol{\rho}}\dot{\mathbf{q}}) + D_{\boldsymbol{\rho}}g \right] \, \mathrm{dt} \tag{5}$$

The derivatives of g in (5) are known for a given objective funcional, but the *sensitivities* $D_{\boldsymbol{\rho}}\mathbf{q}$ and $D_{\boldsymbol{\rho}}\dot{\mathbf{q}}$ have to be computed, solving a set of differential equations (called the tangent linear model) obtained through the derivation of the original equations of motion. If the system is subjected to holonomic constraints collected in vector Φ that are enforced by a penalty method, these original equations take the form:

$$\mathbf{M}\ddot{\mathbf{q}} + D_{\mathbf{q}}\Phi^{\mathrm{T}}\boldsymbol{\alpha}\Phi + \mathbf{Q} = \mathbf{0} \tag{6}$$

M being the mass matrix, $\boldsymbol{\alpha}$ the penalty matrix (typically diagonal), and $\mathbf{Q}(\mathbf{q}, \dot{\mathbf{q}}, t)$ the vector of active forces. If the motion is parametrized with cartesian coordinates of selected points of the system, $\mathbf{M}$ is constant and $\mathbf{Q}$ is free of velocity-dependent gyroscopic terms. Thus, if the system does not contain any physical dissipative device (e.g. dampers), $\mathbf{Q}$ depends only on $\mathbf{q}$ and t.

3.1 Tangent linear model

The tangent linear model is obtained deriving the equations of motion (6) respect to the parameters $\boldsymbol{\rho}$. Under the previous assumptions and denoting by $(\cdot)' = D_{\boldsymbol{\rho}}(\cdot) = \partial(\cdot)/\partial\boldsymbol{\rho}$ to simplify the notation, the equations take the following form:

$$\mathbf{M}\ddot{\mathbf{q}}' + \hat{\mathbf{K}}\mathbf{q}' + \hat{\mathbf{Q}} = \mathbf{0} \tag{7}$$

$\mathbf{q}'$, $\dot{\mathbf{q}}'$ and $\ddot{\mathbf{q}}'$ being the position, velocity and acceleration sensitivities respectively. The terms $\hat{\mathbf{K}}$ and $\hat{\mathbf{Q}}$ are:

$$\hat{\mathbf{Q}} = \mathbf{Q}' + D_{\boldsymbol{q}}\boldsymbol{\Phi}^{T'}\boldsymbol{\alpha}\boldsymbol{\Phi} + D_{\boldsymbol{q}}\boldsymbol{\Phi}^{T}\boldsymbol{\alpha}\boldsymbol{\Phi}' + \mathbf{M}'\ddot{\mathbf{q}} \tag{8}$$

$$\hat{\mathbf{K}} = D_{\boldsymbol{q}}\mathbf{Q} + D_{\boldsymbol{qq}}^{\mathrm{T}}\boldsymbol{\alpha}\boldsymbol{\Phi} + D_{\boldsymbol{q}}\boldsymbol{\Phi}^{T}\boldsymbol{\alpha}D_{\boldsymbol{q}}\boldsymbol{\Phi} \quad , \tag{9}$$

which is a system of linear differential equations that, complemented with the adequate initial conditions, provide de sensitivities $\mathbf{q}'(t)$ and $\dot{\mathbf{q}}'(t)$ provided that the motion $\mathbf{q}(t)$, $\dot{\mathbf{q}}(t)$ and $\ddot{\mathbf{q}}(t)$ are known. In practice, the computation of the sensitivities may be performed after the solution for the dynamics, or simultaneously (step-by-step) with it.

Discussion about optimization in multibody problems and details of alternative techniques for computing sensitivities can be found in [2] and references therein.

3.2 Analytical computation of coefficients

Each component and constraint of the model contributes to the terms in (7). Coefficients associated with simple elements (springs, dampers, etc.) and constraints are typically amenable to an analytical deduction, while coefficients associated with flexible elements are typically obtained with finite-difference techniques. *The relevant proposal of this paper is the analytical deduction of these terms associated with flexible parts, specifically beams. This can be done thanks to the extremely simple formulation of the particular beam model described in the previous section.* The selected parameters of the beam are four: the Young modulus E, the initial cross section a_0, the cross-sectional inertia I and the total initial length L_0. The density has not been considered here since it involves the term $\mathbf{M}'\ddot{\mathbf{q}}$, and the acceleration is very noisy because the dynamics has been solved using a pure penalty method.

The **axial response** is only affected by (a_0, E, L_0), and the derivatives respect these parameters of the elemental axial force (1) are, assuming that the beam is discretized in n identical segments, and so $\lambda_i = r_i n/L_0$:

- Respect to the initial cross-section a_0:

$$D_{a_0}\mathbf{F}_i = D_{\lambda_i} W_i \begin{Bmatrix} \mathbf{e}_i \\ -\mathbf{e}_i \end{Bmatrix} \tag{10}$$

where $D_\lambda W$takes different forms depending of the type of material (nonlinear or linear):

$$D_\lambda W^{nonlin} = \frac{E}{\lambda}\log\lambda \quad , \qquad D_\lambda W^{lin} = E(\lambda - 1) \tag{11}$$

- Respect to the Young modulus E:

$$D_E\mathbf{F}_i = a_0 D_{\lambda_i,E} W_i \begin{Bmatrix} \mathbf{e}_i \\ -\mathbf{e}_i \end{Bmatrix} \;, \quad \text{with} \quad D_{\lambda,E} W_i^{nonlin} = \frac{1}{\lambda}\log\lambda \;, \qquad D_{\lambda_i,E} W_i^{lin} = \lambda - 1 \tag{12}$$

- Respect to the initial length L_0:

$$D_{L_0}\mathbf{F}_i = a_0 (D_{\lambda_i,\lambda_i} W_i)(D_{L_0}\lambda_i) \begin{Bmatrix} \mathbf{e}_i \\ -\mathbf{e}_i \end{Bmatrix} \;, \quad \text{with} \quad D_{\lambda,\lambda} W^{nonlin} = \frac{E}{\lambda^2}(1 - \log\lambda) \;, \qquad D_{\lambda,\lambda} W^{lin} = E \tag{13}$$

y $D_{L_0}\lambda_i = -r_i n / L_0^2$.

The **bending response** is only affected by (E, I, L_0), and the derivatives respect these parameters of the elemental bending force (3) are, assuming again that the beam is discretized in n identical segments, and so $\lambda_i = r_i n/L_0$:

- Respect to the Young modulus E:

$$D_E\mathbf{F}_j = -(D_E\alpha)\phi_j D_{\mathbf{q}_j}\phi_j^{\mathrm{T}} \quad \text{with} \quad D_E\alpha = \frac{1}{E}\boldsymbol{\alpha} \tag{14}$$

- Respect to the cross-sectional inertia I:

$$D_E\mathbf{F}_j = -(D_I\alpha)\phi_j D_{\mathbf{q}_j}\phi_j^{\mathrm{T}} \quad \text{with} \quad D_I\boldsymbol{\alpha} = \frac{1}{I}\boldsymbol{\alpha} \tag{15}$$

- Respect to the initial total length L_0:

$$D_{L_0}\mathbf{F}_j = -(D_{L_0}\alpha)\phi_j (D_{\mathbf{q}_j}\phi_j)^{\mathrm{T}} - \alpha (D_{L_0}\phi_j)(D_{\mathbf{q}_j}\phi_j)^{\mathrm{T}} - \alpha\phi_j (D_{\mathbf{q}_j,L_0}\phi_j)^{\mathrm{T}} \tag{16}$$

that, taking into account:

$$D_{L_0}\alpha = -\frac{5}{L_0}\alpha \;, \quad D_{L_0}\phi_j = \frac{2}{L_0}\phi_j \;, \quad (D_{\mathbf{q}_j,L_0}\phi_j)^{\mathrm{T}} = \frac{1}{L_0}(D_{\mathbf{q}_j}\phi_j)^{\mathrm{T}} \;, \tag{17}$$

result in:

$$D_{L_0}\mathbf{F}_j = -\frac{2\alpha}{L_0}\phi_j (D_{\mathbf{q}_j}\phi_j)^{\mathrm{T}} \tag{18}$$

4 Example: five bar mechanism

Figure 2 shows the initial configuration of the example system, studied by many authors, composed by rigid bars that are released from rest and moves under the sole action of gravity with $g = 9.81$ m/s^2. The masses are $m_{A1} = m_{3B} = 1$ kg, $m_{12} = m23 = 1.5$ kg and the natural lengths of the springs coincide with their lengths in the initial configuration. In order to test the proposed formulation, we replace the rigid bar between the points labelled 1 and 2 by a circular-section beam of the same mass with $a_0 = 1.0628 \cdot 10^{-4}$ m^2, $I = 9.4125 \cdot 10^{-10}$ m^4 made of a material of Young modulus $E = 206.94$ GPa, Poisson modulus $\nu = 0.288$ and density $\rho = 7829$ kg/m^3. The beam is discretized with two segments.

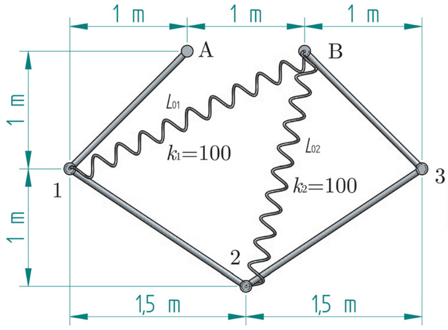

Figure 2. Five-bar mechanism, from [2]

Figure 3 shows the result obtained for the sensibility of the horizontal position of node 1 respect to the Young modulus E of the bar. In the same plot is shown the result obtained with a finite difference scheme, where the simulation is performed twice, one of them with the parameter E sligjtly perturbed (0.1%). With this approach, the sensibility is approximated as the ratio between the increments of the value and the parameter. Apparently, the analytical approach produces considerably larger sensitivities; nevertheless, observe the small values, of the order of 10^{-13}, which is very close to the machine precision in any case.

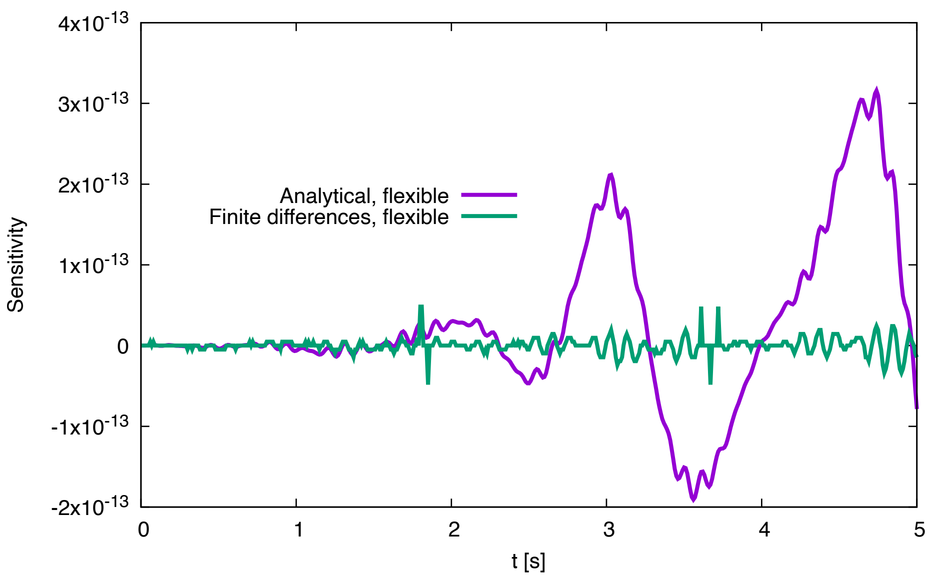

Figure 3. Five-bar mechanism. Sensitivity of the horizontal position of node 1 respect the Young modulus of the bar 12

Figure 4 shows the sensibility of of the horizontal position of node 1 respect to the initial length of the bar 12. The result is compared with the one obtained with finite differences for the flexible and rigid cases.

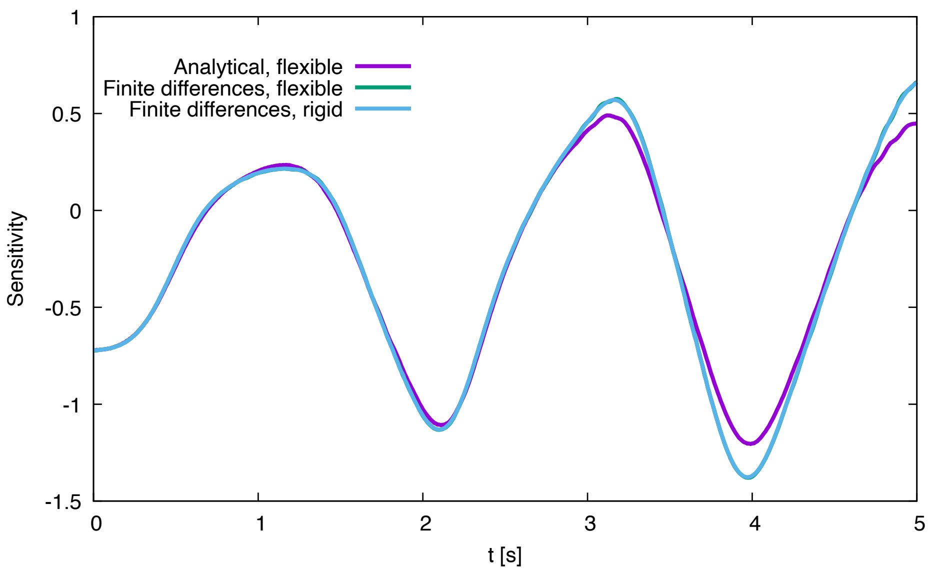

Figure 4. Five-bar mechanism. Sensitivity of the horizontal position of node 1 respect the initial length of bar 12

5 Conclusions

The preliminary results obtained with the analytical approach have the same order of magnitude as the ones obtained with finite differences. For E the sensibilities are larger but very small in any case; this result makes sense, due to the large value of E in the reference model. Nevertheless, we have to investigate this behaviour more deeply, e.g. computing the sensibilities with a smaller reference value for E.

The sensibility respect to the initial length of bar 12 is increasingly larger, compared to the one obtained with finite differences, as the time goes by, and very similar to the rigid model in any case. Again, we have to deeper investigate this effect which is, nevertheless, very small.

In general, a deeper analysis will be performed in a near future, obtaining sensibilities with all the parameters, including a larger number of segments for the beam model. Once the correctness of the approach has been validated, it will be tested with more complex models, including 3D geometries.

Acknowledgements. The authors wish to acknowledge the financial support of the Spanish Ministry of Science and Innovation as part of projects PID2020-120270GB-C22 and PID2020-120270GB-C21 under the names *Metodos de integracion consistentes y elementos finitos no lineales para el analisis de sensibilidad de la dinamica de mecanismos flexibles* and *Diseño y control optimo de sistemas multicuerpo flexibles basado en analisis de sensibilidad de formulaciones con sistema de referencia flotante*.

Authorship statement. The authors hereby confirm that they are the sole liable persons responsible for the authorship of this work, and that all material that has been herein included as part of the present paper is either the property (and authorship) of the authors, or has the permission of the owners to be included here.

References

[1] J. García Orden and J. Cuenca Queipo. A simple shear and torsion-free beam model for multibody dynamics. *Journal of Computational and Nonlinear Dynamics*, vol. 12, n. 5, pp. 1–8, 2017.

[2] Y. S. A. S. C. Daniel; Zhu. Dopico. Direct and adjoint sensitivity analysis of ordinary differential equation multibody formulations. *Journal of Computational and Nonlinear Dynamics*, vol. 10, pp. 011012, 2014.