A Posteriori Error Estimation For Linear Elasticity With Weak Stress Symmetry

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Abstract. This work focuses on the development of a posteriori error estimation for linear elasticity with weak stress symmetry [1]. The procedure is based on the post-processing of the enriched displacement field computed by a mixed finite element formulation, as in [2]. Inspired by [3], the estimation involves two post-processing techniques: averaging the numerical displacement over element interfaces and solving a set of local problems. By applying the Prager-Synge theorem in the context of linear elasticity [4], we are able to develop an estimator with known constants.

Keywords: A posteriori error estimation, Elasticity problem, Adaptive procedure

1 Introduction

The linear elasticity model is important for understanding strain and stress distributions in many engineering applications. The mixed formulations for this kind of problem have gained significant attention and popularity across the years since they can provide more accurate and reliable approximations of stresses and strains [1]. In the context of the mixed formulations for the elasticity problem, we cite [2] where the displacement and the Lagrange multiplier, which weakly imposes symmetry for the stress tensor, are taken in enriched approximation spaces, leading to higher rates of convergence for the divergence of the stress field and for the displacement.

To ensure reliable numerical simulations, it is crucial to quantify and control the errors that arise during the discretization and approximation process. In this context, a posteriori error estimation techniques have emerged as powerful tools for assessing the quality and accuracy of computed solutions. Unlike a priori error estimates, which are derived before the solution process, a posteriori error techniques assess the error based on the computed solution itself. By exploiting information from the numerical solution, these methods can be used to refine the computational mesh or adjust the approximation space, leading to more accurate results and optimal use of computational resources.

This article aims to provide a fully computable a posteriori error estimation for linear elasticity problems based on the reconstructed procedure inspired by the Prager-Synge theorem in the context of elasticity problem [4]. In [5], an a posteriori error estimation applying the reconstruction of the displacement technique is proposed. The procedure follows two main steps: a post-processing of the displacement, when a local projection problem is solved to obtain a discontinuous displacement with enhanced accuracy of order, followed by an averaging operator in order to get a continuous displacement.

The procedure adopted in this work can be seen as an extension of the method developed in [3] for the multiscale mixed method applied to the Darcy's problem. In [3] the approximation for the flux variable is already an equilibrated global H(div)-conforming flux approximation, so the methodology for the error estimation only requires a potential reconstruction. As in [2], the enriched approximation space is applied, and it is possible to obtain higher rates of convergence for the divergence of the stress field and for the displacement, so no post-processing of the displacement is required. Therefore, in our context, the smoother displacement field will be obtained applying a smooth procedure following by a local Dirichlet problem.

The paper is organized as follows: in Section 2 the notations and the model problem associated with linear elasticity weak symmetry are introduced. In Section 3, an exposition of the enriched mixed approximation finite element spaces is provided. Last, in Section 4, we present fundamental findings related to a posteriori error estimation and a new procedure is presented.

2 Preliminaries

Let $\Omega \subset \mathbb{R}^2$ be a connected polyhedral domain filled by a linearly elastic body. Throughout the text, for a region $D \subseteq \Omega$, we denote by $\mathbf{n}^D$ the external unit vector normal to ∂D. Both scalar Hilbert spaces $L^2(D)$ and $H^s(D)$ have their usual meanings and norms. Associated vector or matrix-valued spaces $L^2(D, \mathbb{E})$, and $H^s(D, \mathbb{E})$ shall be considered, where $\mathbb{E}$ represents either $\mathbb{R}^2$, $\mathbb{M} = \mathbb{R}^{2\times 2}$, symmetric $\mathbb{S} \subset \mathbb{M}$, or skew symmetric $\mathbb{K} \subset \mathbb{M}$ tensors. These spaces inherit the corresponding norms associated with the inner products in $L^2(D)$ and $H^s(D)$. The space $H(\mathrm{div}, D)$ denotes the square-integrable vector functions, taking values in $\mathbb{R}^2$, for which the divergence is also square-integrable. Similarly, we consider tensor functions in $H(\mathrm{div}, D, \mathbb{M})$. These spaces inherit the corresponding norms associated with the inner products $(\cdot,\cdot)_D$ in $L^2(D)$, $L^2(D, \mathbb{R}^2)$, and $L^2(D, \mathbb{M})$. The notation $\langle\cdot,\cdot\rangle_{\partial D}$ is adopted to define the duality pairing between the vector spaces

$$H^{1/2}(\partial D, \mathbb{R}^2) \;=\; \left\{\underline{\mu} = \underline{u}|_{\partial D},\ \underline{u} \in H^1(D, \mathbb{R}^2)\right\} \quad \text{and} \tag{1}$$

$$H^{-1/2}(\partial D, \mathbb{R}^2) \;=\; \left\{\underline{\mu} = \underline{\underline{\tau}}\,\mathbf{n}^D,\ \underline{\underline{\tau}} \in H(\mathrm{div}, D, \mathbb{M})\right\} \tag{2}$$

Gradient (∇) and divergence ($\nabla\cdot$) operators applied to scalar and vector fields have the usual meaning. For tensor fields $\underline{\underline{\tau}}$, the divergence ($\underline{\nabla}\cdot$) is the vector field obtained by taking the divergence of each row $\underline{\tau}_i$. The *Green's Formula* for $\underline{\underline{\tau}} \in H(\mathrm{div}, D, \mathbb{M})$ and $\underline{v} \in H^1(D, \mathbb{R}^2)$ reads

$$(\underline{\nabla}\cdot\ \underline{\underline{\tau}}, \underline{v})_D = -(\underline{\underline{\tau}}, \underline{\underline{\varepsilon}}(\underline{v}))_D - (\underline{\underline{\tau}}, \underline{\underline{\eta}}(\underline{v}))_D + \left\langle \underline{\underline{\tau}}\,\mathbf{n}^D, \underline{v}\right\rangle_{\partial D} \tag{3}$$

where, for a vector $\underline{u}$, the *strain* and *rotation* tensors are given by

$$\underline{\underline{\varepsilon}}(\underline{u}) = \frac{\nabla\underline{u} + \nabla\underline{u}^T}{2}, \quad \underline{\underline{\eta}}(\underline{u}) = \frac{\nabla\underline{u} - \nabla\underline{u}^T}{2}. \tag{4}$$

For a vector $\underline{v} = (v_1, v_2)$ the curl operator gives $\underline{\nabla} \times \underline{v} = (\partial_2 v_1, -\partial_1 v_1)$, and for a tensor with rows $\underline{\tau}_i$, $\underline{\underline{\nabla}} \times \underline{\underline{\tau}} = (\underline{\nabla} \times \underline{\tau}_i)$.

The equations of static elasticity in Hellinger-Reissner form determine fields for the stress $\underline{\underline{\sigma}}$ and the displacement $\underline{u}$ such that,

$$\underline{\nabla}\cdot\underline{\underline{\sigma}} \;=\; \underline{f} \ \text{ in } \Omega, \tag{5}$$

$$\underline{\underline{\sigma}} \;=\; \underline{\underline{A}}\,\underline{\underline{\varepsilon}}(\underline{u}) \ \text{ in } \Omega, \tag{6}$$

$$\underline{u} \;=\; \underline{g}, \ \text{ on } \partial\Omega_D, \tag{7}$$

$$\underline{\underline{\sigma}}\,\eta \;=\; \underline{0}, \ \text{ on } \partial\Omega_N, \tag{8}$$

where the functions $\underline{g} \in H^{\frac{1}{2}}(\partial\Omega_D, \mathbb{R}^2)$ and $\underline{f} \in L^2(\Omega, \mathbb{R}^2)$ are given Dirichlet boundary data and body force, $\partial\Omega_D$ with positive measure. The material properties are described by the constitutive tensor $\underline{\underline{A}} = \underline{\underline{A}}(\underline{x})$, which is a self-adjoint, bounded, and uniformly positive definite linear operator acting from $\mathbb{S}$ to $\mathbb{S}$. We assume that $\underline{\underline{A}}$ can be extended to an operator from $\mathbb{M}$ to $\mathbb{M}$ with the same properties. In particular, in the case of homogeneous and isotropic body,

$$\underline{\underline{A}}\,\underline{\underline{\varepsilon}} = 2\mu\underline{\underline{\varepsilon}} + \lambda\ \mathrm{tr}(\underline{\underline{\varepsilon}})\underline{\underline{I}}, \tag{9}$$

where λ and μ represent the Lamé parameters and $\underline{\underline{I}}$ the 2×2 identity matrix.

This elasticity model problem admits an equivalent expression, without assuming a priori stress symmetry, by replacing the original constitutive equation $\underline{\underline{\sigma}} = \underline{\underline{A}}\ \underline{\underline{\varepsilon}}(\underline{u})$ by $\underline{\underline{A}}^{-1}\underline{\underline{\sigma}} = \nabla\underline{u} - \underline{\underline{\eta}}(\underline{u})$, using the relation $\underline{\underline{\varepsilon}}(\underline{u}) = \nabla\underline{u} - \underline{\underline{\eta}}(\underline{u})$. A new equation $\underline{\underline{\sigma}} - \underline{\underline{\sigma}}^T = \underline{\underline{0}}$ is then added to enforce the desired stress symmetry, and consequently a new rotation variable $\underline{\underline{q}} = \underline{\underline{\eta}}(\underline{u}) \in L^2(\Omega, \mathbb{K})$ is introduced, acting as a Lagrange multiplier to enforce the symmetry constraint on the stress tensor.

Under this point of view, the mixed formulation with weakly imposed stress symmetry searches for $(\underline{\underline{\sigma}}, \underline{u}, \underline{\underline{q}}) \in H(\text{div}, \Omega, \mathbb{M}) \times L^2(\Omega, \mathbb{R}^d) \times L^2(\Omega, \mathbb{K})$ satisfying $\underline{\underline{\sigma}}\, \underline{n}^{\Omega}|_{\Omega_N} = \underline{0}$, and

$$(\underline{\underline{A}}^{-1}\, \underline{\underline{\sigma}}, \underline{\underline{\tau}}) + (\underline{u}, \underline{\nabla} \cdot \underline{\underline{\tau}}) + (\underline{\underline{q}}, \underline{\underline{\tau}}) = \langle \underline{\underline{\tau}}\, \mathbf{n}^{\Omega}, \underline{g} \rangle_{\partial\Omega_D}, \quad \forall \underline{\underline{\tau}} \in H(\text{div}, \Omega, \mathbb{M}), \tag{10}$$

$$(\underline{\nabla} \cdot \underline{\underline{\sigma}}, \underline{v}) = (\underline{f}, \underline{v}), \qquad \forall \underline{v} \in L^2(\Omega, \mathbb{R}^2), \tag{11}$$

$$(\underline{\underline{\sigma}}, \underline{\underline{\beta}}) = 0, \qquad \forall \underline{\underline{\beta}} \in L^2(\Omega, \mathbb{K}). \tag{12}$$

This kind of variational formulation typically appears in minimization problems with constraints [6]. For this model problem, there is a constraint for the accounting of the divergence equation (11), and displacement plays the role of the corresponding Lagrange multiplier. The other multiplier is $\underline{\underline{q}}$, used for the weak enforcement of stress symmetry in (12). According to Brezzi's theory [6], the variational formulation (10)-(12) is well posed, but for its FE discretization the approximations spaces of each field cannot be chosen independently one from the other, i.e., they should be compatible, meaning that some stability (inf-sup) conditions are mandatory.

3 Weak Symmetry (WS) FE approximation

For the finite element discretization of problem (10)-(12), a partition of the domain Ω is taken as $\mathcal{T}_h$ composed of elements K (quadrilateral or triangular elements). The local polynomial spaces are: $P_k(K)$, the polynomial of degree at most k, $Q_{k_1,k_2}(K)$, the polynomial of maximum degrees k_1 and k_2 in each variable, and $\tilde{P}_k(K)$, the homogeneous polynomial of degree k. We also consider the local vector space $\mathbf{V}(K)$ decomposed as $\mathbf{V}(K) = \mathbf{V}_k^{\partial}(K) \oplus \mathbf{V}_k^{\circ}(K)$, where $\mathbf{V}_k^{\partial}$ denotes the space of edge functions and $\mathbf{V}_k^{\circ}$ denotes the space of internal functions, and the space $S(K, \mathbb{M})$ of tensors having each row in $\mathbf{V}(K)$.

The mappings of the finite element spaces from the reference element $\hat{K}$ to the geometric element K are defined by:

- For a scalar function, $v = \hat{v} \circ F_K^{-1}$, where $F_K : \hat{K} \to K$ is an invertible map;
- For a vector function, $\underline{q} = \mathbb{F}_K^{\text{div}} \hat{\underline{q}} = \mathbb{F}_K \left[\frac{1}{\mathbf{J}_K} DF_K\right]$, where DF_K denotes the Jacobian of F_K, $\mathbf{J}_K = |\det(DF_K)|$, and $\mathbb{F}_K^{\text{div}}$ is the Piola transformation;
- For a tensor function, $\underline{\underline{q}} = \mathbb{F}_K^{\text{div}} \hat{\underline{\underline{q}}}$, applying the Piola transformation on each row of $\hat{\underline{q}}$.

The finite element spaces $\mathscr{S} \times \mathscr{U} \times \mathscr{Q} \subset H(\text{div}, \Omega, \mathbb{M}) \times L^2(\Omega, \mathbb{R}^2) \times L^2(\Omega, \mathbb{K})$ are defined by

$$\mathscr{S} = \{\underline{\underline{\tau}} \in H(\text{div}, \Omega, \mathbb{M}) : \underline{\underline{\tau}}|_K \in S(K, \mathbb{M}), \forall K \in \mathcal{T}_h\}, \tag{13}$$

$$\mathscr{U} = \{\underline{v} \in L^2(\Omega, \mathbb{R}^2) : \underline{v}|_K \in U(K), \forall K \in \mathcal{T}_h\}, \tag{14}$$

$$\mathscr{Q} = \{\underline{\underline{\beta}} \in L^2(\Omega, \mathbb{K}) : \underline{\underline{\beta}}|_K \in Q(K, \mathbb{K}), \forall K \in \mathcal{T}_h\}, \tag{15}$$

where the local spaces $S(K, \mathbb{M})$ and $U(K)$ are taken to have in their rows divergence-consistent FE pairs $V(K) \times P(K)$ that are usually applied for flux and potential approximations in mixed formulations of Poisson problems. The FE space $\mathscr{Q}$ needs to be taken in order to guarantee the following compatibility constraint on the FE pair $\mathscr{S} \times \mathscr{Q}$: there must exist a vector space $\mathscr{W} \subset H^1(K, \mathbb{R}^2)$ such that $\mathscr{W} \times \mathscr{Q}$ is a Stokes-compatible pair, and $\underline{\underline{\nabla}} \times \mathscr{W} \subset \mathscr{S}$. For different cases of stability for the FE pair $\mathscr{S} \times \mathscr{Q}$ we cite [7].

For the stability of the saddle-point problem (13)-(15) we cite [8] and references therein.

Finally, the discrete problem can be stated as: find $(\tilde{\underline{\underline{\sigma}}}, \tilde{\underline{u}}, \tilde{\underline{\underline{q}}}) \in \mathscr{S} \times \mathscr{U} \times \mathscr{Q}$ such that

$$(\underline{\underline{A}}^{-1}\, \tilde{\underline{\underline{\sigma}}}, \underline{\underline{\tau}}) + (\tilde{\underline{u}}, \underline{\nabla} \cdot \underline{\underline{\tau}}) + (\tilde{\underline{\underline{q}}}, \underline{\underline{\tau}}) = \langle \underline{\underline{\tau}}\, \mathbf{n}^{\Omega}, \underline{g} \rangle_{\partial\Omega_D}, \quad \forall \underline{\underline{\tau}} \in \mathscr{S}, \tag{16}$$

$$(\underline{\nabla} \cdot \tilde{\underline{\underline{\sigma}}}, \underline{v}) = (\underline{f}, \underline{v}), \qquad \forall \underline{v} \in \mathscr{U}, \tag{17}$$

$$(\tilde{\underline{\underline{\sigma}}}, \underline{\underline{\beta}}) = 0, \qquad \forall \underline{\underline{\beta}} \in \mathscr{Q}. \tag{18}$$

The enriched approximation spaces used to solve the discrete problem (16) - (18) are of type $\mathcal{BDM}_k^{+}$ and $\mathcal{BDM}_k^{++}$, for triangular elements, and $\mathcal{RT}_{[k]}^{+}$, for quadrilateral elements. The rate of convergence in the L^2-norm for these spaces using affine meshes are described in Table 1. For a complete description and further details, we refer to [2].

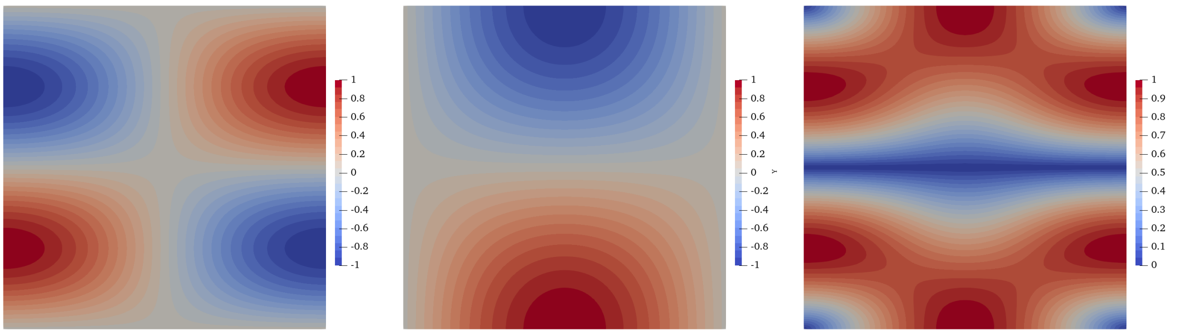
3.1 Numerical WS Problem

In this section, we describe a model problem for elasticity with weak symmetry by applying the enriched mixed method described in the previous section. Let $\Omega = [0, 1]^2$. The load function $\underline{g}$ is chosen such that the

Geometry	Space	$\underline{\underline{\sigma}}$	$\underline{u}$	$\nabla \cdot \underline{\underline{\sigma}}$	$\underline{q}$
Triangular	$\mathcal{BDM}_k^+$	$k+1$	$k+1$	$k+1$	$k+1$
	$\mathcal{BDM}_k^{++}$	$k+1$	$k+2$	$k+2$	$k+1$
Quadrilateral	$\mathcal{RT}_{[k]}^+$	$k+1$	$k+2$	$k+2$	$k+1$

Table 1. Rate of convergence in the L^2-norm for enriched finite element spaces for the mixed formulation of linear elasticity with weakly imposed stress symmetry.

exact displacement is $\underline{u} = \begin{pmatrix} \cos(\pi x)\sin(2\pi y) \\ \cos(\pi y)\sin(\pi x) \end{pmatrix}$,with the Lamé parameters $\lambda = 123$ and $\mu = 79.3$. The exact displacement components are illustrated in Figure 1(a) and Figure 1(b). The L^2-norm is illustrated in Figure 1(c).



(a) u_x. (b) u_y. (c) $||\underline{u}||_{L^2(\Omega,\mathbb{R}^2)}$.

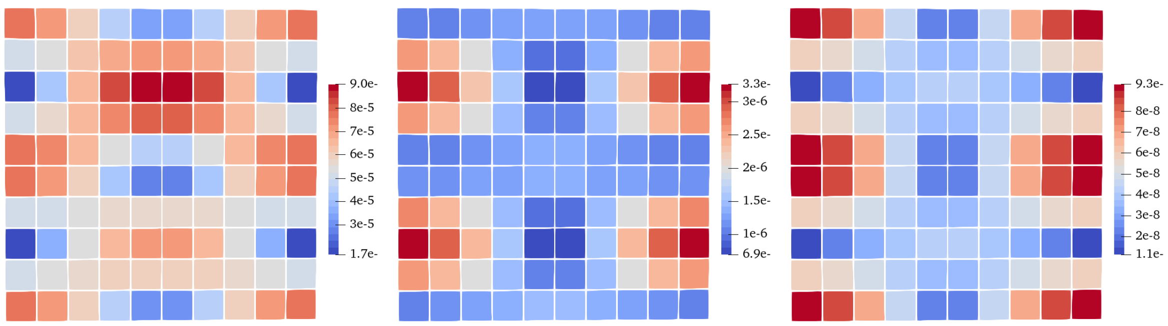
Figure 1. Exact solution for an elasticity model problem with weak symmetry.

The approximation of the problem is done considering a rectangular refinement mesh and the approximation space of type $\mathcal{RT}_{[k]}^+$, with $k \in \{1, 2, 3\}$. The results for histories of convergence are illustrated in Figure 3. Observe that the rate of convergence agrees with the ones summarized in Table 1.

Finally, we compute the exact error in each element of a regular mesh with $h = 1/10$ and for different polynomial orders $k = 1, 2$ and 3. The results are presented in Figure 2.

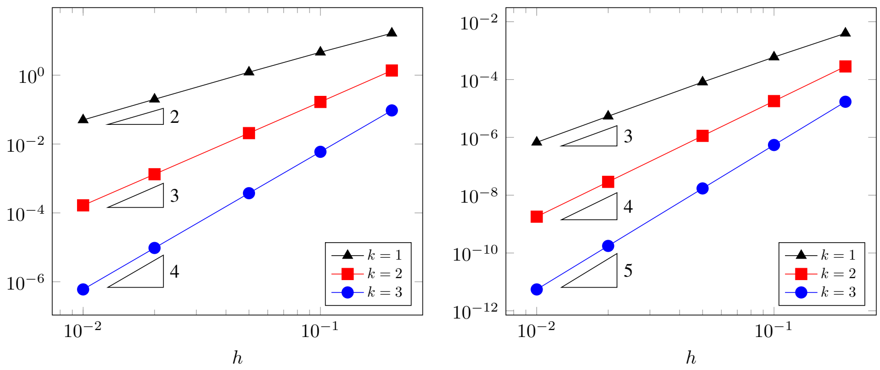
4 A Posteriori Error Estimation

A posteriori error estimation techniques play a crucial role in improving the accuracy and efficiency of numerical solutions for a large variety of problems, with particular significance in elasticity problems. In this context, a posteriori error estimation involves estimating the error in quantities of interest, such as displacement, stress, or



(a) $k = 1$ (b) $k = 2$ (c) $k = 3$

Figure 2. Element displacement error.



(a) Stress error (b) Displacement error

Figure 3. Histories of convergence for displacement and stress tensor using mixed approximation space $\mathcal{RT}^{+}_{[k]}$ with $k \in \{1, 2, 3\}$.

strain fields, by utilizing information obtained from the numerical solution itself. This approach enables the identification of regions with higher error, guiding adaptive refinement strategies and improving the overall accuracy of the solution.

In this work, we are inspired by the Prager-Synge Theorem (PS) [9], which relates the error in the strain energy norm to the error in the displacement field. Applying the PS Theorem, we propose to extend the results presented in [3] for the linear elasticity problem. More specifically, we incorporate the recovery technique for the displacement field to obtain an upper bound limited error for the stress tensor field. Given the significance of the PS Theorem, we present a version of the theorem originally introduced in [9] and adapted in [5] in terms of the weighted norm $\|\underline{\underline{\tau}}\|^2_{\underline{\underline{A}}^{-1}} = (\underline{\underline{A}}^{-1}\, \underline{\underline{\tau}}, \underline{\underline{\tau}})$.

Theorem 1. *Given a stress tensor* $\underline{\underline{\sigma}}^* \in H(\mathrm{div}, \Omega, \mathbb{M})$ *such that* $\underline{\nabla} \cdot \underline{\underline{\sigma}}^* + \underline{f} = \underline{0}$, $\underline{\underline{\sigma}}^*\, \mathbf{n}^{\Omega}|_{\Omega_N} = \underline{0}$, *and* $\underline{u}^* \in H^1(\Omega, \mathbb{R}^2)$, *with* $\underline{u}^*|_{\partial\Omega_D} = \underline{0}$, *then*

$$\|\underline{\underline{\sigma}} - \underline{\underline{\sigma}}^*\|^2_{\underline{\underline{A}}^{-1}} + \|\underline{\underline{\sigma}} - \underline{\underline{A}}\, \underline{\underline{\varepsilon}}(\underline{u}^*)\|^2_{\underline{\underline{A}}^{-1}} = \|\underline{\underline{\sigma}}^* - \underline{\underline{A}}\, \underline{\underline{\varepsilon}}(\underline{u}^*)\|^2_{\underline{\underline{A}}^{-1}}. \tag{19}$$

Employing finite element approximations for the mixed elasticity problem represented by equations (16)-(18), it can be noticed that the stress tensor is already within the space $H(\mathrm{div}, \Omega, \mathbb{M})$. As a consequence, it becomes possible to access an upper bound for the stress tensor approximation when a reconstructed displacement variable $\underline{u} \in H^1(\Omega, \mathbb{R}^2)$ is obtained. In this case,

$$\|\underline{\underline{\sigma}} - \underline{\underline{\sigma}}^*\|^2_{\underline{\underline{A}}^{-1}} \leq \|\underline{\underline{\sigma}}^* - \underline{\underline{A}}\, \underline{\underline{\varepsilon}}(\underline{u}^*)\|^2_{\underline{\underline{A}}^{-1}}. \tag{20}$$

The reconstructed displacement proposed in [5] involves two main steps: post-processing of the displacement, when a local projection problem is solved to obtain a discontinuous displacement with enhanced accuracy of order $k+1$, followed by an averaging operator in order to get a continuous displacement $\underline{u}^* \in H^1(\Omega, \mathbb{R}^2)$ with $\underline{u}^*|_{\partial\Omega_D} = \underline{0}$.

In this work, as the displacement field obtained by the enriched mixed approximation space is already of order $k+1$, no post-processing is necessary. Therefore, our purpose to recovery the displacement in $H^1(\Omega, \mathbb{R}^2)$ is to adapt the procedure described in [3] for the MHM-H(div)-$\mathcal{E}_\gamma$ method in the context of linear elasticity with weakly imposed stress symmetry, as described in the next section.

4.1 Procedure for displacement reconstruction

Given an approximation $(\underline{\underline{\tilde{\sigma}}}, \underline{\tilde{u}}, \underline{\underline{\tilde{q}}})$, solution of the discrete problem (16)-(18), we denote each component of the displacement $\underline{\tilde{u}} \in \mathscr{U} \subset L^2(\Omega, \mathbb{R}^2)$ by $\tilde{u}_i$ with $i = 1, 2$. Also, we denote by Λ the piecewise polynomial space defined over the inter-element edges $F_{i,j} = \partial K^i \cap \partial K^j$ for all $K^i, K^j \in \mathcal{T}_h$. As each component of the displacement $\underline{\tilde{u}}$ is in $L^2(\Omega)$, the procedure to recompose a displacement in $H^1(\Omega, \mathbb{R}^2)$ will follow the same procedure described in [3] for mixed formulation for Darcy's problem and recalled below.

Inter-element smoothing procedure The purpose is to construct a function $\underline{\mu} \in \Lambda$ from the approximate displacement $\underline{\tilde{u}}$.

The inter-element smoothing procedure for 2D approximations occurs in two steps: weighted averaging of the face values over the edges, and weighted averaging of the edge values at nodes. The averaged solution values over the edges are represented by 2D and 1D discontinuous elements of appropriate polynomial order.

1. Average over edges: for an edge e not included in the $\partial\Omega_D$ set, build the patch $\mathcal{T}(e)$ formed by all the interfaces $F_{i,j} = \partial K^i \cap \partial K^j$ having e as one of their edges. Then, associate with $\underline{\mu}|_{F_{i,j}}$, and update $\underline{\mu}|_e \in \Lambda|_e$ by fitting
$$\int_e \left[\frac{1}{2}\left(\underline{\tilde{u}}|_{K^i} + \underline{\tilde{u}}|_{K^j}\right) - \underline{\mu}|_e\right] \underline{v}\, ds, \quad \forall\, \underline{v} \in \Lambda|_e. \tag{21}$$

2. Average over vertices: for a vertex $\mathbf{a}$ not included in $\partial\Omega_D$, set the patch $\mathcal{T}(\mathbf{a})$ of the $N_{\mathbf{a}}$ edges e having $\boldsymbol{a}$ as one of their vertices. Then, update the value $\underline{\mu}(\mathbf{a})$ by the average
$$\underline{\mu}(\mathbf{a}) \longleftarrow \frac{1}{N_{\mathbf{a}}} \sum_{e \in \mathcal{T}(\mathbf{a})} \underline{\mu}|_e(\mathbf{a}). \tag{22}$$

Finally, update $\underline{\mu}|_{F_{i,j}}(\mathbf{x}) \longleftarrow \underline{\mu}_{\mathbf{a}}(\mathbf{x}) + \underline{\mu}_e(\mathbf{x})$, represented in terms of hierarchical vertex and edge shape functions associated with $F_{i,j}$, by usual operations used in the assembly algorithms for hp H^1-conforming FE spaces [10]. This is done in a sequence of steps, starting with vertices, then throughout edges and the face $F_{i,j}$ itself. The vertex component $\underline{\mu}_{\mathbf{a}}(\mathbf{x})$ is computed by first-order Lagrange interpolation of the new values $\underline{\mu}(\mathbf{a})$ of Step 2 at the vertices of $F_{i,j}$. Next, by incorporating the vertex term, the edge component $\underline{\mu}_e(\mathbf{x})$, vanishing on the vertices, is obtained by updating the edge averages of Step 1.

Solving local Dirichlet problem Let $\underline{\mu} \in \Lambda$ be given by the inter-element smoothing procedure described above, and set $\underline{\underline{\tilde{\boldsymbol{\sigma}}}}_i := \underline{\underline{\tilde{\sigma}}}|_{K^i}$. The reconstructed displacement $\underline{u}^* \in H^1(\Omega, \mathbb{R}^2)$ is defined as $\underline{u}^*|_{K^i} = \underline{u}^*_i$, where the functions $\underline{u}^*_i$ are obtained by solving $H^1(\Omega, \mathbb{R}^2)$-conforming Galerkin FE formulations of local problems in K^i. The solvers are for FE spaces $\tilde{\mathscr{U}}(K^i) \subset H^1(K^i, \mathbb{R}^2) \cap \mathscr{U}(K^i)$ and defined by: Find $\underline{u}^* \in \tilde{\mathscr{U}}(K^i)$ such that
$$(\underline{\underline{\varepsilon}}(\underline{u}^*_i), \underline{\underline{\varepsilon}}(\underline{v}))_{K^i} = (\underline{\underline{A}}^{-1}\underline{\underline{\tilde{\boldsymbol{\sigma}}}}_i, \underline{\underline{\varepsilon}}(\underline{v}))_{K^i} \qquad \forall\, \underline{v} \in H^1(K^i, \mathbb{R}^2) \cap \mathscr{U}(K^i) \tag{23}$$
$$\underline{u}^*|_{\partial K_i} = \underline{\mu}|_{\partial K_i} \text{ in } \partial K^i \tag{24}$$

By the constraints, $\underline{u}^*_i = \underline{\mu}$ on $\partial\Omega_i$ the continuity of $\underline{u}^*$ over interfaces is satisfied (i.e. $\underline{u}^* \in H^1(\Omega, \mathbb{R}^2)$). Moreover, since by definition $\underline{\mu}|_{\Gamma_D} = \underline{\tilde{u}}$, we conclude that $\underline{u}^*$ is indeed a displacement reconstruction.

Finally, with a reconstructed displacement field, we can obtain the following upper bound for the stress error.

Theorem 2. *Let* $(\underline{\underline{\tilde{\sigma}}}, \underline{\tilde{u}}, \underline{\underline{\tilde{q}}})$ *be an approximate solution for the elasticity problem* (16)-(18) *obtained by the enriched mixed method. If* $\underline{u}^* \in H^1(\Omega, \mathbb{R}^2) \cap \mathscr{U}$ *is a reconstructed displacement then*
$$\|\underline{\underline{\sigma}} - \underline{\underline{\tilde{\sigma}}}\|^2_{\underline{\underline{A}}^{-1}} \leq \|\underline{\underline{\tilde{\sigma}}} - \underline{\underline{A}}\,\underline{\underline{\varepsilon}}(\underline{u}^*)\|^2_{\underline{\underline{A}}^{-1}} + osc(\underline{f})^2 + osc(\underline{g})^2, \tag{25}$$
where $osc(\underline{f}) = C\left(\sum_{K^i \in \mathcal{T}_h} h^2_{K^i} \|\underline{f} - \Pi\,\underline{f}\|^2_{L^2(\Omega,\mathbb{R}^2)}\right)^{1/2}$ *and* $osc(\underline{g}) = C\left(\sum_{E \subset \partial\Omega_N} h_\Gamma \|\underline{g} - \tilde{\Pi}\,\underline{g}\|^2_{L^2(E,\mathbb{R}^2)}\right)^{1/2}$ *with* Π *and* $\tilde{\Pi}$ *denoting the usual* L^2 *projection operators.*

Proof. See [5], Theorem 6. □

5 Conclusions

In this paper, an a posteriori error estimator is presented for the linear elasticity problem solved using enrichment approximation space within a mixed formulation. The procedure to derive the error estimator is based on the Prager-Synge Theorem and includes displacement recovery, which is considered essential for its accuracy. This work extends the procedure adopted in a previous study [3] to the context of linear elasticity problems.

To demonstrate the potential and practical application of the proposed approach in mesh adaptivity, numerical examples are currently being prepared. These examples will showcase how the error estimator performs under different scenarios.

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