Dimensional reduction of probability spaces via Sobol' indices applied to composite laminates optimal design

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Abstract. It is proposed the application of Sobol' indices as importance measures for the dimensionality reduction of probability spaces, in probabilistic reliability assessment. The indices are computed approximately using a local approximation of the response functionals, via an adjoint method. As a numerical example, it is considered the Reliability-based Robust Design Optimization (RBRDO) of a composite shell structure. The problem is defined as the bi-objective minimization problem of the total weight and the determinant of the variance-covariance matrix of the structural response functionals, subject to a deterministic displacement constraint and a probabilistic stress constraint. The goal is to study how the dimensionality reduction in reliability assessment affects the RBRDO of composite laminate structures. The results show that, from a total of 16 random mechanical properties, only 5 to 8 are important, while explaining at least 99.7% of the uncertainty. It is achieved a drastic reduction in computing times between 2 to 12 times faster, in reliability assessment, and 2 times faster in the overall RBDO of composite laminate structures.

Keywords: Sobol' indices, importance analysis, reliability assessment, composites optimal design.

1 Introduction

The design of composite laminate structures is particularly difficult, due to the uncertainty associated with the main design parameters and material properties. Optimal designs are solutions above the average and are therefore more sensitive to uncertainty. In this work, uncertainty is quantified according to the concepts of robustness and reliability. Robustness is a measure of the variability associated with the performance and/or the response of a structural system. An optimization methodology promoting a systematic analysis and control of structural variability is called Robust Design Optimization (RDO) [1]. Here, reliability is understood as a probabilistic measure of structural failure, due to random uncertainty. An optimization methodology promoting the systematic analysis and control of probabilistic failure is called Reliability-based Design Optimization (RBDO) [2]. Recently, the combination of both RDO and RBDO has been explored in the design optimization of composite laminate structures, however with increased computational effort, mostly due to highly dimensional probability spaces. Such design methodology is called Reliability-based Robust Design Optimization (RBRDO) [3]. An analytical dimensional reduction criterion, based on the approximate solution of Sobol' indices, was recently proposed [4]. When a first-order approximation of said indices is considered only an adjoint system of equations is required to be solved. In problems where the majority of the probability density of random variables is concentrated around the mean-values, the proposed first-order (local) approximation is capable of capturing/quantifying the importance of the random variables relatively to the system response functionals, with relatively low computational cost, when compared to most dimensionality reduction methods employing global sampling methods [4].

The aim of this work is to implement an efficient dimensional reduction method, based on the importance analysis theory of Sobol', to systematically determine the important random variables in the inner loop of reliability assessment of each design solution found during the RBRDO of composite structures.

2 Sobol' Importance Analysis

A proper definition of *importance* is necessary. Here, *importance* is a measure of the reduction in the variance of a stochastic function if one or more random variables are fixed on a given value. Let $\mathbf{x} \in \Omega$ be a generic vector of statistically independent random variables defined in a probability space $(\Omega, \mathcal{F}, P)$ and $g: \Omega \to \mathbb{R}$ a square-integrable stochastic function over Ω w.r.t. the probability measure P. We establish the partial variances [5]:

$$V_i = \mathrm{var}\big(\mathrm{E}(g(\mathbf{x})|x_i)\big), \quad V_{i,j} = \mathrm{var}\Big(\mathrm{E}\big(g(\mathbf{x})|x_i, x_j\big)\Big) - V_i - V_j, \quad \dots \tag{1}$$

where $\mathrm{E}(\cdot\,|x_i)$ is the conditional expectation with respect to all random variables except x_i, and so forth for all possible combinations of random variables. Then, Sobol' indices are uniquely defined as the ratio between each partial variance and the total variance of g [5]:

$$S_i = \frac{\mathrm{var}\big(\mathrm{E}(g(\mathbf{x})|x_i)\big)}{\mathrm{var}\big(g(\mathbf{x})\big)}, \quad S_{i,j} = \frac{\Big(\mathrm{var}\Big(\mathrm{E}\big(g(\mathbf{x})|x_i, x_j\big)\Big) - V_i - V_j\Big)}{\mathrm{var}\big(g(\mathbf{x})\big)}, \quad \dots \tag{2}$$

such that:

$$\sum_{i=1}^{N} S_i + \sum_{i<j}^{N} S_{i,j} + \dots + S_{1,2,\dots,N} = 1 \tag{3}$$

where S_i is the first-order Sobol' index, w.r.t. x_i, $S_{i,j}$ is the second-order Sobol' index, w.r.t. (x_i, x_j), and so on.

2.1 Analytical approximate solution by Propagation of Moments

Based on a multilinear interpretation of the theory of propagation of moments [4], consider the following. Let $I = \{i_1, \dots, 1_N\}$ be a set of N indices, in ascending order, and $I' \subseteq I$ with cardinality $\alpha_{I'} \leq k \leq N$, containing combinations of $\alpha_{I'}$ ordered indices. Thus, the partial variances in (2) become:

$$V_{I'} \simeq \left(\frac{\partial^{\alpha_{I'}} g}{\prod_{i \in I'} \partial x_i}\right)^2_{\boldsymbol{\mu}_\mathbf{x}} \prod\nolimits_{i \in I'} \mathrm{Var}(x_i) \tag{4}$$

and the total variance becomes $\mathrm{var}\big(g(\mathbf{x})\big) \simeq \sum_{I'} V_{I'}$. Finally, the Sobol' indices in (2) are approximated by:

$$S_{I'} \simeq \frac{V_{I'}}{\sum_{I'} V_{I'}} \in [0,1] \tag{5}$$

Thus, a random variable x_{i_k}, with $i_k \in I$ and $k \leq N$, is said to be important if the sum of all Sobol' indices w.r.t. x_{i_k} is greater than or equal to a non-negative scalar ε. That is:

$$S_{i_k} + \sum\nolimits_{j \in I} S_{i_k,j} + \sum\nolimits_{j<l \in I} S_{i_k,j,l} + \dots + S_{i_1,\dots,i_N} \geq \varepsilon \tag{6}$$

Of particular interest, for the current case study, is the approximation by a first-order Taylor polynomial, for which only the main effect indices are non-null. From (5), Sobol' indices become:

$$S_{x_i} = \left(\frac{\partial g}{\partial x_i}\right)^2_{\boldsymbol{\mu}_\mathbf{x}} \frac{\mathrm{var}(x_i)}{\mathrm{var}\big(g(\mathbf{x})\big)} \tag{7}$$

3 RBRDO of composite Laminates

Let $\mathbf{z} \in \Omega_\mathrm{Z}$ be a vector of random design variables, whose vector of mean- values $\boldsymbol{\mu}_\mathbf{z}$ contains the design variables of the problem. Additionally, consider a vector $\boldsymbol{\pi} \in \Omega_\Pi$ of random structural parameters, whose mean values $\boldsymbol{\mu}_{\boldsymbol{\pi}}$ are inputs of the problem. The vectors of standard deviations $\boldsymbol{\sigma}_\mathbf{z}$ and $\boldsymbol{\sigma}_{\boldsymbol{\pi}}$ are constant and are an input of the problem, as well. Thus, a functional $g_i(\mathbf{z}, \boldsymbol{\pi}) : \Omega_\mathrm{Z} \times \Omega_\Pi \to \mathbb{R}$ is called the i-th stochastic response functional of the structural system and $g_i(\boldsymbol{\mu}_{\mathbf{z}}, \boldsymbol{\mu}_{\boldsymbol{\pi}})$ its deterministic realization.

The system response functionals are defined in terms of the displacement and the stress responses of composite laminate structures. The structural analysis of composite laminate structures is based on a displacement formulation of the Finite Element Method (FEM), in particular the shell finite element model developed by Ahmad et al. [6]. Generically, let $U \subseteq \mathbb{R}^M$ be the state space of state variables $\mathbf{u}$ (displacements) and $\Omega = \Omega_Z \times \Omega_\Pi \subseteq \mathbb{R}^N$ a sample space. In this work, it is considered the linear equilibrium of structures, under static loading conditions, defined by a mapping $\Psi: U \times \Omega \to U$, such that the following (implicit) state equation:

$$\Psi(\mathbf{u}(\mathbf{z}, \boldsymbol{\pi}), \mathbf{z}, \boldsymbol{\pi}) = 0 \Leftrightarrow \mathbf{Ku} - \mathbf{f} = \mathbf{0} \tag{8}$$

holds, where $\mathbf{K}(\mathbf{z}, \boldsymbol{\pi})$ is the stiffness matrix and $\mathbf{f}$ the load vector. The applied geometric discretization of the physical system [6] implies calculating the displacement and stress fields in specific points identified by the coordinates (e, p, k), where e is the element number, p is the ply number and k is the integration point. Hence, the deterministic response functional $g_1(\boldsymbol{\mu}_{\mathbf{z}}, \boldsymbol{\mu}_{\boldsymbol{\pi}})$, associated with the critical displacement, is given by:

$$g_1(\boldsymbol{\mu}_{\mathbf{z}}, \boldsymbol{\mu}_{\boldsymbol{\pi}}) = \frac{\max u_{i(e,k)}}{u_a} - 1 \tag{9}$$

where u_a is the allowable displacement value, related with structural design. The stress state is characterized by a scalar measure of structural integrity. To account for the coupling between failure modes, the Tsai-Wu quadratic failure criterion is applied, returning the parameter R known as the Tsai number ($R > 1$ meaning safety) [7]. The deterministic response functional associated with the critical stress state of the structure is given by:

$$g_2(\boldsymbol{\mu}_{\mathbf{z}}, \boldsymbol{\mu}_{\boldsymbol{\pi}}) = \frac{\min R_{(e,p,k)}}{R_a} - 1 \tag{10}$$

where R_a is the allowable Tsai number.

3.1 Robustness Assessment

As a robustness measure, it is considered the determinant of the variance–covariance matrix, $\mathbf{C}$, associated with the stochastic displacement and stress functionals, whose components are given by:

$$\operatorname{cov}(g_i, g_j) \simeq \sum_{k=1}^{N_z} \left(\frac{\partial g_i}{\partial z_k}\right)_{\boldsymbol{\mu}_{\mathbf{z}}} \left(\frac{\partial g_j}{\partial z_k}\right)_{\boldsymbol{\mu}_{\mathbf{z}}} \sigma_{z_k}^2 + \sum_{k=1}^{N_\pi} \left(\frac{\partial g_i}{\partial \pi_k}\right)_{\boldsymbol{\mu}_{\boldsymbol{\pi}}} \left(\frac{\partial g_j}{\partial \pi_k}\right)_{\boldsymbol{\mu}_{\boldsymbol{\pi}}} \sigma_{\pi_k}^2 \tag{11}$$

based on a first-order Taylor approximation of the response functionals and with all partial derivatives calculated via an adjoint method.

3.2 Reliability Assessment

In structural reliability, a functional $g : U \times \Omega \to \mathbb{R}$, with both implicit and explicit dependence on $(\mathbf{z}, \boldsymbol{\pi})$ is called *limit-state function*; and a probability space $(\Omega, \mathcal{F}, P)$, with $\mathcal{F} = \{\emptyset, D_f. D_s, \Omega\}$, is called *uncertainty space*, where $D_f = \{(\mathbf{z}, \boldsymbol{\pi}) \in \Omega : g(\mathbf{z}, \boldsymbol{\pi}) < 0\}$ and $D_s = \overline{D_f}$ are disjoint subsets of Ω, called the *failure space* and the *safety space*, respectively. The structural *probability of failure* is here estimated indirectly by the identification of the point on the failure surface, $g(\mathbf{z}, \boldsymbol{\pi}) = 0$, with the greatest probability density. In the space of independent standard normal random variables $\mathbf{y} \sim N(\mathbf{0}, \mathbf{1})$, the identification of such point is a minimization problem:

$$\begin{aligned} \min_{\mathbf{y}} \quad & \|\mathbf{y}\| \\ \text{subject to:} \quad & G(\mathbf{y}) = 0 \end{aligned} \tag{12}$$

where $\mathbf{y} = T(\mathbf{z}, \boldsymbol{\pi})$, $T: \Omega \to Y$ is an invertible transformation and $G(\mathbf{y}) = g(T^{-1}(\mathbf{y}))$. The solution to the problem is called the *most probable failure point* (MPP), $\mathbf{y}_{\mathbf{MPP}}$, and $\beta_{HL} = \|\mathbf{y}_{\mathbf{MPP}}\|$ is the Hasofer-Lind reliability index [8], such that the probability of failure is approximately equal to [2,8]:

$$p_f \simeq \Phi(-\beta_{HL}) \tag{13}$$

3.3 Reliability-based Robust Design Optimization problem

The proposed RBRDO problem of composite laminate structures is, thus, defined as follows:

$$\begin{aligned} \min_{\boldsymbol{\mu}_{\mathbf{z}} \in \Omega_Z} \quad & \mathbf{f}(\boldsymbol{\mu}_{\mathbf{z}}, \boldsymbol{\mu}_{\boldsymbol{\pi}}) = \left(W(\boldsymbol{\mu}_{\mathbf{z}}, \boldsymbol{\mu}_{\boldsymbol{\pi}}), \det \mathbf{C}_{\boldsymbol{\varphi}}\right) \\ \text{subject to:} \quad & g_1(\boldsymbol{\mu}_{\mathbf{z}}, \boldsymbol{\mu}_{\boldsymbol{\pi}}) \leq 0 \\ & \beta_{HL}(\boldsymbol{\mu}_{\mathbf{z}}, \boldsymbol{\mu}_{\boldsymbol{\pi}}) \geq \beta_a \\ & \mu_{z_i}^{low} \leq \mu_{z_i} \leq \mu_{z_i}^{up} \end{aligned} \tag{14}$$

where W is the total structural weight, β_a is the allowable reliability index and $\mu_{z_i}^{up}$ and $\mu_{z_i}^{low}$ are upper and lower boundaries of the design variables, respectively.

4 Numerical application

In the present study, four sources of uncertainty are considered and divided into two sets of variables: random design variables, $\mathbf{z}$, and random parameters, $\boldsymbol{\pi}$. They are organized as follows: material properties, $\mathbf{m} \subseteq \boldsymbol{\pi}$, point loads, $\mathbf{p} \subseteq \boldsymbol{\pi}$, ply angle of the laminates, $\theta \subseteq \mathbf{z}$, and laminate thicknesses, $\mathbf{h} \subseteq \mathbf{z}$. Depending on the uncertainty quantification measure (robustness or reliability), the sources of uncertainty differ. In robustness assessment, uncertainty is propagated through the random design variables $\mathbf{z}$ and the random parameters $\boldsymbol{\pi}$. The mean values of the random design variables $\boldsymbol{\mu}_{\mathbf{z}}$ are the design variables of the RBRDO problem, thus varying between design solutions. In reliability assessment, the uncertainty of the system is propagated only through the mechanical properties $\mathbf{m}$. Furthermore, it is assumed that the random material properties follow $\mathbf{m} \sim N(\boldsymbol{\mu}_{\mathbf{m}}, \boldsymbol{\sigma}_{\mathbf{m}})$. Regarding the Sobol' importance analysis, the proposed dimensional reduction criterion is applied solely to the reliability assessment inner-cycle of each design solution generated during the optimization process, because it is the most expensive part of the design optimization process. In structural design problems, lower-order approximations are often preferred. For the current problem, it is considered a first-order approximation (see (7)) and a threshold value of $\varepsilon = 0.001$. Hence, a random mechanical property m_i, for $i = 1, \dots, N_m$, is considered important if $S_{m_i} \geq 0.001$.

The algorithm used to solve the bi-objective RBRDO problem is the Bi-level Dominance Multi-Objective Genetic Algorithm (MOGA-2D) [9]. The algorithm searches the design space to find multiple Pareto-optimal solutions in parallel, using two simultaneous populations, namely, the short population, $\mathbf{SP}^t$, and the enlarged population $\mathbf{EP}^t$, at each generation t. It performs using the concept of local constrain-dominance at the $\mathbf{SP}^t$, storing the nondominated solutions into the $\mathbf{EP}^t$.

The Hasofer-Lind reliability index problem, in (12), is solved by the Hybrid micro-Genetic Algorithm (HmGA), with deterministic operators developed specifically for equality-constraint handling, namely the Genetic Repair and the Region Zooming mechanisms. The algorithm further relies on the dynamics between highly elitist and highly disruptive stochastic operators and works on a mixed real-binary genotype space [10]. The global convergence, with probability 1, to the global optimum reliability index was mathematically demonstrated in [11].

4.1 Cylindrical shell structure

As a numerical example, it is considered a clamped cylindrical laminated shell structure. Nine vertical loads of $P = 11.5$ kN are applied on the side AB. The structure is divided into four laminates (two elements per laminate), each with five layers and stacking sequence $[+\theta/-\theta/0/-\theta/+\theta]$. The ply angle θ is referenced to the x-axis. Variables h_i, $i = 1, \dots, 4$, denote the laminates' thickness, as show in Figure 1. A composite material built with the carbon/epoxy system denoted T300/N5208 [12] is considered. The elastic constants of the orthotropic ply are the longitudinal elastic modulus E_1, the transversal elastic modulus E_2, the in-plane shear modulus G_{12} and the in-plain Poisson's ratio v_{12}. The ply strength properties are the longitudinal tensile strength X, the longitudinal compression strength X' , the transversal tensile strength Y, the transversal compression strength Y' and the shear strength S. The mean-values of the material properties are in Table 1. Overall, the vector of random design variables, **z**, contains four laminate thickness variables and one ply angle variable (common to all laminates).The vector of random parameters, $\boldsymbol{\pi}$, contains nine point loads, and sixteen material properties (four per laminate) grouped in vector $\mathbf{m} = \left(E_1^j, E_2^j, Y^j, S^j\right)$, where $j = 1, \dots, 4$ is the laminate number. Finally, the allowable critical displacement is $u_a = 8$ cm, the allowable critical Tsai number is $R_a = 1$ and the allowable reliability index is $\beta_a = 3.0$. The standard deviations of the random variables are set to 6% of the mean values.

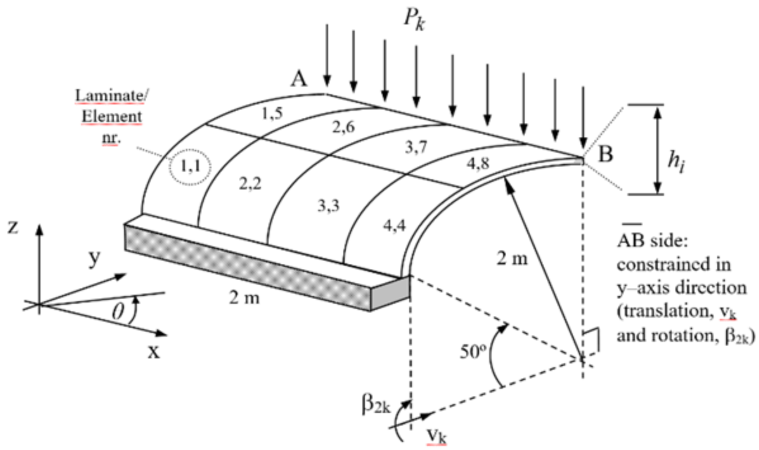

Figure 1. Cylindrical shell structure with point loads distribution

Table 1. Average properties of the carbon/epoxy T300/N5208 [12].

E_1 [GPa]	E_2 [GPa]	G_{12} [GPa]	ν_{12}
181.00	10.30	7.17	0.28
$X; X'$[MPa]	**$Y; Y'$[MPa]**	**S [MPa]**	**ρ [kg/m^3]**
1500; 1500	40; 246	68	1600

4.2 Results

The goal of this numerical application is to study how the dimensional reduction of the uncertainty spaces, associated with reliability assessment, affects the RBRDO of composite laminate structures. Conclusions are drawn by comparison with the results obtained with without any dimensional reduction. To distinguish between the two cases, consider RBRDO^0 (for $\varepsilon = 0$) and RBRDO$^\varepsilon$ (for $\varepsilon = 0.001$). As a first comment, notice that the first-order partial derivatives of the structural response functionals are calculated only once, for each design solution found by the MOGA-2D algorithm.

Figure 2 shows the non-dominated fronts of the problem, obtained with both the RBRDO^0 and RBRDO$^\varepsilon$ models, after 300 generations. Overall, both models converged to the same region of the objective space, indicating that the dimensionality reduction didn't affect the design optimization process. However, there is a considerable difference in the computing times necessary to achieve both fronts: the RBRDO^0 model took about 27 h to complete the design process, while the RBRDO$^\varepsilon$ took about 14.5 h, under the same working conditions (Intel(R) Core(TM) i7-6700 CPU @ 3.40Ghz). It represents an improvement of the computing times of 1.8 times.

The computing times of reliability assessment were also subject to comparison. To avoid a biased analysis, 20 solutions of the non-dominated front obtained by the RBRDO^0 model (no dimensional reduction) were selected, and their reliability index calculated independently, with and without dimensional reduction. On Figure 3, it is seen that, without dimensional reduction (on the left), the HmGA took between 4 to 12 minutes to compute each reliability index. On the right, it is seen that, under the same convergence criteria, the HmGA took 4 to 6 minutes to converge (black bars), while achieving better predictions of the reliability index. However, by relaxing the convergence criteria of the HmGA (see [4, 10] for details), it is found that the algorithm converges to similar results as those obtained without dimensional reduction in just 0.5 to 2.5 minutes, being 2 to 12 times faster.

Overall, for all the design solutions obtained by the RBRDO$^{\varepsilon}$ model, during the entire optimization process, it is found that only 5 to 8 random material properties are classified as *important*. It represents a reduction of at least half of the number of dimensions, in reliability assessment. Choosing four different non-dominated solutions of the RBRDO$^{\varepsilon}$ model (identified with black circles in Figure 2), it is found that the important material properties alone explain at least 99.77% of the uncertainty in the vicinity of the mean-values. In other words, at least 99.77% of the uncertainty is confined to the important dimensions and variations in the values of the non-important material properties will have a negligible effect on the value of the reliability index. Table 2 shows the values of Sobol' indices of the important random variables, for each of the selected non-dominated solutions.

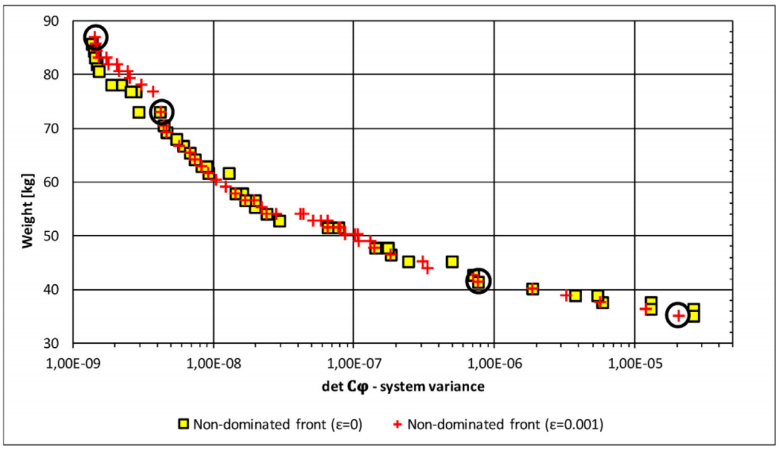

Figure 2. Non-dominated fronts of the RBRDO problem, obtained with the RBRDO^{0} and RBRDO$^{\varepsilon}$ models.

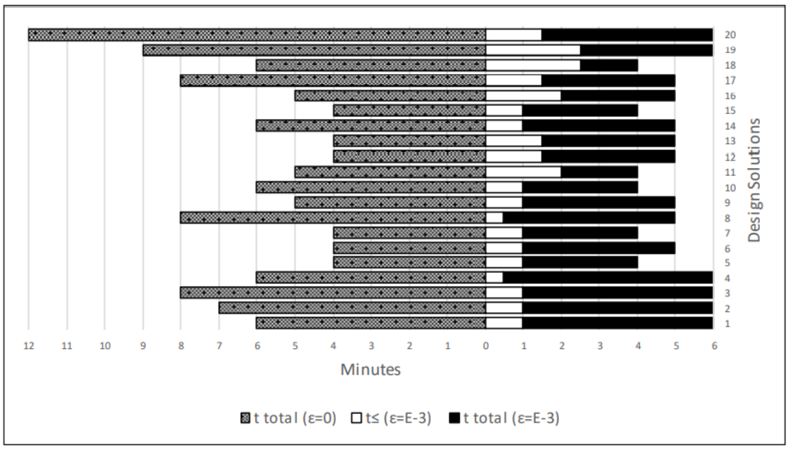

Figure 3. Computing times of reliability assessment, with and without dimensional reduction, for 20 nondominated solutions of the RBRDO^{0} model.

(A)	$S_{E_{11}}$ 0.1567	$S_{E_{21}}$ 0.0072	S_{Y_1} 0.7911	$S_{E_{12}}$ 0,0122	$S_{E_{14}}$ 0.0017	$S_{E_{24}}$ 0.0288		Σ 0.9977	
(B)	$S_{E_{11}}$ 0.0031	$S_{E_{12}}$ 0.0799	$S_{E_{22}}$ 0.8697	S_{Y_2} 0.0021	S_{S_2} 0.0031		Σ 0.9992		
(C)	$S_{E_{11}}$ 0.0499	$S_{E_{21}}$ 0.0157	S_{Y_1} 0.8661	S_{S_1} 0.0106	$S_{E_{12}}$ 0.0334	$S_{E_{13}}$ 0.00561	$S_{E_{23}}$ 0.00142	$S_{E_{24}}$ 0.01694	Σ 0.9997
(D)	$S_{E_{11}}$ 0.0072	$S_{E_{21}}$ 0.0599	S_{Y_1} 0.9101	$S_{E_{12}}$ 0.0054	$S_{E_{13}}$ 0.0143	$S_{E_{23}}$ 0.0015	$S_{E_{24}}$ 0.0009		Σ 0.9993

Table 2. Sobol' indices of selected non-dominated solutions obtained by the RBRDO$^{\varepsilon}$ model.

5 Conclusions

An analytical dimensional reduction criterion of the uncertainty spaces associated with reliability assessment, as proposed earlier in [4], is considered. The criterion is based on the approximate solution of Sobol' indices, based on multilinear Taylor polynomial approximations of the limit-state function. A first-order approximation was considered and the partial derivatives were calculated by an adjoint method, only requiring an adjoint system of equilibrium equations to be solved. The criterion was then applied to a Reliability-based Design Optimization (RBRDO) of composite laminate structures, solved exclusively with evolutionary algorithms (EAs), aiming to reduce the number of random mechanical properties considered in reliability assessment. As a numerical example, it was considered a composite shell, with four laminates. The results show that, from a total of sixteen random mechanical properties considered in reliability assessment, only five to eight are considered as important. In fact, the important random variables explain at least 99.7% of the uncertainty space, which is also the likelihood of the reduced uncertainty space containing the true MPP. Thus, the number of possible genetic combinations explored in the reliability assessment inner-cycle is exponentially smaller. Consequently, the proposed dimensional reduction allowed the RBRDO problem to be solved almost twice as fast. At the same time, the numerical results were identical to those obtained without any dimensional reduction. Therefore, demonstrating the goodness of the proposed method.

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