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A discontinuous and nonlinear multiscale method for solving convection-dominated problems

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Abstract. This work presents a discontinuous and nonlinear multiscale method for solving problems dominated by convection. The method introduces a nonlinear artificial diffusion term at both scales of discretization while employing a discontinuous framework solely at the coarse scale. The micro scale is approximated using bubble functions, enabling efficient and accurate representation of the solution behavior. This approach aims to improve the accuracy and stability of numerical simulations for convection-dominated phenomena. Convection-dominated problems pose challenges in accurately resolving steep gradients and rapid variations in the solution. Conventional numerical methods often encounter problems related to numerical stability when solving this type of problem. In order to overcome these limitations, the proposed numerical scheme combines the benefits of nonlinear artificial diffusion, the discontinuous framework, and the use of bubble functions. To validate the effectiveness of the new method, some numerical experiments were conducted on convection-dominated problems of varying complexity. The results demonstrate that the multiscale method outperforms traditional approaches in accurately capturing the solution behavior, particularly in regions with sharp gradients.

Keywords: DG methods, Discontinuous Dynamic Diffusion method, Convection-Diffusion equation, Bubble function.

1 Introduction

When dealing with convection-dominated problems, classical numerical methods may present numerical instabilities if the diffusion coefficient is significantly smaller than the advection coefficient. This situation is characterized by a physical problem with a high Péclet number, leading to a hyperbolic-like behavior in its solution. Consequently, these conventional numerical schemes, such as the Galerkin finite element (FEM) and the finite difference methods, become inadequate in capturing the resulting boundary layer phenomenon. The mesh refinement strategy is not suitable in this context because it leads to a considerable increase in computational effort.

A practical way to overcome the weaknesses of the classical finite element method, for this type of problem, is to resort to stabilized methods. These methods introduce artificial diffusion into the numerical model so that the resulting formulation is consistent and provides stable numerical solutions. As examples of stabilized formulations to solve convection-dominated transport problems, we mention the Streamline Upwind Petrov-Galerkin (SUPG) in Brooks and Hughes [1]; the Galerkin-Least Square method (GLS) in Hughes et al. [2]; the Continuous Interior Penalty (CIP) method in Burman and Hansbo [3], and the Local Projection Stabilization (LPS) in Barrenechea et al. [4]. All these methods are linear and provide globally stable numerical solutions. However, localized oscillations may remain in regions of high gradients. The elimination of these local oscillations can be obtained through discontinuity capture schemes or Spurious Oscillations at Layers Diminishing (SOLD) methods, as presented in John and Knobloch [5]. In general, SOLD methods consist of adding non-linear operators to linear stabilized methods, in order to reduce the localized oscillations.

All of the previously cited methods for dealing with convection-dominated problems involve stabilized formulations within the Continuous Galerkin framework. Another important class of methods to solve this problem is based on the Discontinuous Galerkin (DG) methodology. DG methods exhibit favorable stability characteristics in pure advection problems, similar to linear stabilized methods, as cited in Johnson and Pitkaranta [6]. Additionally, they offer advantages in robustness, especially when applied to first-order differential operators associated with hyperbolic equations, as described in Hughes et al. [7].

Arruda et al. [8] developed two discontinuous Galerkin formulations within the framework of nonlinear two-scale methods for solving advection-diffusion-reaction equations. In Arruda et al. [9] the authors present the discontinuous Dynamic Diffusion (DD) method. These discontinuous methods consider a two-level discretization of the approximation space so that two nested grids must be built. The variational formulation is based on the multiscale methodology, where the problem is partitioned into two parts: macro and micro, as described in Hughes [10] and Hughes et al. [11]. Furthermore, nonlinear artificial diffusion operators are added to the numerical model. The discontinuous and nonlinear DD method was rewritten in the continuous setting using bubble functions to described the micro scale in Santos et al. [12], Valli et al. [13] and Santos et al. [14], resulting in a method with good stability and convergence properties.

In this work we present a discontinuous and nonlinear two-scale method for solving convection-dominated problems. Different from the discontinuous methods presented in Arruda et al. [8, 9], this method uses bubble functions to discretize the micro scale. Besides, the discontinuous methodology is applied only on the macro scale of the discretization. We compare its computational results with a DG method, in the solution of two convection-dominated problems with internal and external layers.

The remainder of this work is organized as follows. In section 2, we briefly address the mathematical model and discontinuous Galerkin formulation. Section 3 is devoted to the description of our proposed discontinuous and nonlinear multiscale method. The numerical experiments are conducted in section 4. Finally, we conclude this paper in section 5.

2 Discontinuous Galerkin formulation

Let $\Omega \in \mathbb{R}^2$ be a bounded open domain with Lipschitz boundary Γ. We consider a convection-diffusion-reaction problem with Dirichlet boundary conditions

$$\begin{aligned} -\kappa\Delta u + \boldsymbol{\beta}\cdot\nabla u + \sigma u &= f \text{ in } \Omega, \\ u &= g \text{ on } \Gamma, \end{aligned} \tag{1}$$

where $\kappa > 0$ is the diffusion coefficient, $\boldsymbol{\beta} \in W^{1,\infty}(\Omega)^d$ is the divergence-free velocity field, $\sigma \in L^\infty(\Omega)$ is the reaction coefficient, $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma_d)$. We also define the inflow boundary $\Gamma_- = \{x \in \Gamma;\ \boldsymbol{\beta}(x)\cdot\mathbf{n}(x) < 0\}$, where $\mathbf{n}(x)$ denotes the unit outward normal vector to Γ at $x \in \Gamma$.

Let $\mathcal{T}_h = \{K\}$ be a triangulation of the domain Ω. The boundary ∂K of $K \in \mathcal{T}_h$ is composed of three edges. We denote $\mathcal{E}_h$ the set of all edges in $\mathcal{T}_h$, $\mathcal{E}_h^0$ and $\mathcal{E}_h^\Gamma$ are the internals and boundaries edges, respectively. The length of an edge e is represented by h_e and $h = \max_{e\in\mathcal{E}_h}\{h_e\}$. Let φ be a scalar piecewise smooth function on $\mathcal{T}_h$. We define the jump and average over an edge e as $[\![\varphi]\!] = \varphi^1\mathbf{n}^1 + \varphi^2\mathbf{n}^2$ and $\{\varphi\} = \frac{1}{2}(\varphi^1 + \varphi^2)$, respectively, where $\mathbf{n}^i$ is the outward unit normal vector of K_i on e with $\varphi^i = \varphi|_{K_i}$, and K_i are the elements sharing the edge. To a vector function $\boldsymbol{\tau}$, those definitions are give by $[\![\boldsymbol{\tau}]\!] = \boldsymbol{\tau}^1\cdot\mathbf{n}^1 + \boldsymbol{\tau}^2\cdot\mathbf{n}^2$ and $\{\boldsymbol{\tau}\} = \frac{1}{2}(\boldsymbol{\tau}^1 + \boldsymbol{\tau}^2)$, respectively.

The space of discontinuous piecewise linear functions on Ω is defined as

$$V_h = \{v \in L^2(\Omega):\ v|_K \in P_1(K),\ \forall K \in \mathcal{T}_h\}, \tag{2}$$

where $P_1(K)$ denotes the space of linear polynomials on element K.

A discontinuous Galerkin formulation for solving eq. (1) is given as follow: find $u_h \in V_h$ such that

$$B_{DG}(u_h, v_h) = F_{DG}(v_h), \quad \forall v_h \in V_h, \tag{3}$$

where

$$
\begin{aligned}
B_{DG}(u_h, v_h) = & \sum_{K\in\mathcal{T}_h} (\kappa\nabla u_h, \nabla v_h)_K + \sum_{K\in\mathcal{T}_h} (\boldsymbol{\beta}\cdot\nabla u_h + \sigma u_h, v_h)_K \\
& - \kappa \sum_{e\in\mathcal{E}_h^0} \langle\{\nabla u_h\}, [\![v_h]\!]\rangle_e + \epsilon_0\kappa \sum_{e\in\mathcal{E}_h^0} \langle[\![u_h]\!], \{\nabla v_h\}\rangle_e + \kappa\eta_0 h^{-1} \sum_{e\in\mathcal{E}_h^0} \langle[\![u_h]\!], [\![v_h]\!]\rangle_e \\
& - \sum_{e\in\mathcal{E}_h^{0-}} \langle\boldsymbol{\beta}\cdot[\![u_h]\!], v_h\rangle_e - \sum_{e\in\mathcal{E}_h^{\Gamma-}} \langle(\boldsymbol{\beta}\cdot\mathbf{n})u_h, v_h\rangle_e \\
& - \kappa \sum_{e\in\mathcal{E}_h^\Gamma} \langle\nabla u_h\cdot\mathbf{n}, v_h\rangle_e + \epsilon_0\kappa \sum_{e\in\mathcal{E}_h^\Gamma} \langle u_h, \nabla v_h\cdot\mathbf{n}\rangle_e + \kappa\eta_\Gamma h^{-1} \sum_{e\in\mathcal{E}_h^\Gamma} \langle u_h, v_h\rangle_e,
\end{aligned} \tag{4}
$$

$$
F_{DG}(v_h) = \sum_{K\in\mathcal{T}_h} (f, v_h)_K + \epsilon_0\kappa \sum_{e\in\mathcal{E}_h^\Gamma} \langle g, \nabla v_h\cdot\mathbf{n}\rangle_e + \kappa\eta_\Gamma h^{-1} \sum_{e\in\mathcal{E}_h^\Gamma} \langle g, v_h\rangle_e - \sum_{e\in\mathcal{E}_h^{\Gamma-}} \langle(\boldsymbol{\beta}\cdot\mathbf{n})g, v_h\rangle_e. \tag{5}
$$

In eq. (4) and eq. (5),

$$
(u, v)_K = \int_K uv\, d\boldsymbol{x}, \qquad \langle u, v\rangle_e = \int_e uv\, ds,
$$

$\epsilon_0 = \{-1, 0, 1\}$, η_0 and η_Γ are the interior and the boundary edges penalty parameters, respectively. This formulation with $\epsilon_0 = 1$ was presented in Houston et al. [15]. Here, we use $\epsilon_0 = -1$, the Symmetric Interior Penalty Galerkin (SIPG) method, described in Arnold [16].

3 A discontinuous Dynamic Diffusion formulation

This formulation is a discontinuous and nonlinear two-scale method, where the discontinuous methodology is applied only on the macro or coarse scale of the discretization. The micro or fine scale is discretized using bubble functions. Furthermore, a nonlinear artificial diffusion operator is added to both discretization scales. The method has its origins in the works presented in Valli et al. [13] and Arruda et al. [9].

The macro scale is discretized by the coarse space, V_h, defined as in eq. (2). This standard finite element space is enriched with bubble functions through space

$$
V_b = \{w \in H^1(\Omega) :\ w|_K \in H_0^1(K),\ \forall K \in \mathcal{T}_h\}, \tag{6}
$$

called fine (or bubble) space. Here, we use the simple cubic polynomial function $\varphi_b = 27N_1^K N_2^K N_3^K$, where $N_j^K = N_j^K(x, y)$ are the basis function on the V_h space. The enriched space is represented as a direct sum of V_h and V_b, that is,

$$
V_E = V_h \oplus V_b. \tag{7}
$$

The method, named here by BDD (discontinuous DD method with bubble functions), consists of finding $u_E = u_h + u_b \in V_E$, with $u_h \in V_h$ and $u_b \in V_b$, such that

$$
B_{DG}(u_E, v_E) + D(u_h; u_E, v_E) = F_{DG}(v_E), \quad \forall v_E \in V_E, \tag{8}
$$

where $v_E = v_h + v_b$, with $v_h \in V_h$, $v_b \in V_b$. In addition, $B_{DG}(\cdot,\cdot)$ and $F_{DG}(\cdot)$ are the bilinear and linear operators, respectively, of the discontinuous Galerkin method defined in eq. (3). The nonlinear artificial diffusion operator, $D(\cdot;\cdot,\cdot)$, is defined as

$$
D(u_h; u_E, v_E) = \sum_{K\in\mathcal{T}_h} \int_K \xi(u_h)\boldsymbol{\nabla} u_E \cdot \boldsymbol{\nabla} v_E\, d\boldsymbol{x}, \tag{9}
$$

where

$$
\xi(u_h) = \begin{cases} \mu(h)\frac{|R(u_h)|}{\|\boldsymbol{\nabla} u_h\|}, & \text{if} \quad \|\boldsymbol{\nabla} u_h\| > tol_\xi, \\ 0, & \text{otherwise.} \end{cases} \tag{10}
$$

In eq. (10), $\mu(h) = \dfrac{1}{2}\sqrt{|K|}$ over the outflow boundary and $\mu(h) = 2\sqrt{|K|}$ otherwise, stands for the subgrid characteristic length, where $|K|$ is the area of the element K,

$$
R(u_h) = -\kappa\Delta u_h + \boldsymbol{\beta}\cdot\boldsymbol{\nabla} u_h + \sigma u_h - f
$$

is the residual of the coarse solution on element K, and tol_ξ is a positive number small enough to avoid division by zero. Here we set $tol_\xi = 10^{-6}$.

The BDD method is solved by using a fixed point iterative procedure defined as: given u_h^n, we find u_h^{n+1} satisfying

$$B_{DG}(u_E^{n+1}, v_E) + D(u_h^n; u_E^{n+1}, v_E) = F_{DG}(v_E), \quad \forall v_E \in V_E, \tag{11}$$

where the initial solution, u_h^0, is the zero solution. To improve the convergence of the iterative process in the step $n+1$ we use the relaxation scheme present in Santos and Almeida [17],

$$\xi^*(u_h^n) = \omega\xi(u_h^n) + (1-\omega)\xi(u_h^{n-1}),$$

with $\omega = 0.5$. In this case, $\xi^*(\cdot)$ is used in eq. (11) instead of $\xi(\cdot)$.

The restriction of the linearized operator D on each element K can be written as

$$D(u_h^n; u_E^{n+1}, v_E)\Big|_K = \int_K \xi^*(u_h^n)\boldsymbol{\nabla} u_h^{n+1} \cdot \boldsymbol{\nabla} v_h \, d\boldsymbol{x} + \int_K \xi^*(u_h^n)\boldsymbol{\nabla} u_b^{n+1} \cdot \boldsymbol{\nabla} v_b \, d\boldsymbol{x},$$

since

$$\int_K \xi^*(u_h^n)\boldsymbol{\nabla} u_h^{n+1} \cdot \boldsymbol{\nabla} v_b \, d\boldsymbol{x} = \int_K \xi^*(u_h^n)\boldsymbol{\nabla} u_b^{n+1} \cdot \boldsymbol{\nabla} v_h \, d\boldsymbol{x} = 0.$$

Thus, the two-scale decomposition of eq. (11), obtained by testing this equation with $v_h \in V_h$ and $v_b \in V_b$, results in two problems,

$$B_{DG}(u_h^{n+1}, v_h) + B_{DG}(u_b^{n+1}, v_h) + \sum_{K\in\mathcal{T}_h} \int_K \xi^*(u_h^n)\boldsymbol{\nabla} u_h^{n+1} \cdot \boldsymbol{\nabla} v_h \, d\boldsymbol{x} = F_{DG}(v_h), \quad \forall v_h \in V_h, \tag{12}$$

$$B_{DG}(u_h^{n+1}, v_b) + B_{DG}(u_b^{n+1}, v_b) + \sum_{K\in\mathcal{T}_h} \int_K \xi^*(u_h^n)\boldsymbol{\nabla} u_b^{n+1} \cdot \boldsymbol{\nabla} v_b \, d\boldsymbol{x} = F_{DG}(v_b), \quad \forall v_b \in V_b, \tag{13}$$

where $B_{DG}(u_h^{n+1}, v_h)$ is given by eq. (4), $F_{DG}(v_h)$ is given by eq. (5),

$$\begin{aligned}
B_{DG}(u_b^{n+1}, v_h) &= \sum_{K\in\mathcal{T}_h} (\kappa\nabla u_b^{n+1}, \nabla v_h)_K + \sum_{K\in\mathcal{T}_h} (\boldsymbol{\beta}\cdot\nabla u_b^{n+1} + \sigma u_b^{n+1}, v_h)_K, \\
B_{DG}(u_h^{n+1}, v_b) &= \sum_{K\in\mathcal{T}_h} (\kappa\nabla u_h^{n+1}, \nabla v_b)_K + \sum_{K\in\mathcal{T}_h} (\boldsymbol{\beta}\cdot\nabla u_h^{n+1} + \sigma u_h^{n+1}, v_b)_K, \\
B_{DG}(u_b^{n+1}, v_b) &= \sum_{K\in\mathcal{T}_h} (\kappa\nabla u_b^{n+1}, \nabla v_b)_K + \sum_{K\in\mathcal{T}_h} (\boldsymbol{\beta}\cdot\nabla u_b^{n+1} + \sigma u_b^{n+1}, v_b)_K, \\
F_{DG}(v_b) &= \sum_{K\in\mathcal{T}_h} (f, v_b)_K,
\end{aligned}$$

since the terms associated with the fine scales at the element boundaries vanishes.

The local algebraic equation system associated with problems in eq. (12) and eq. (13), on each element K, may be partitioned as

$$\begin{bmatrix} A_{hh} & A_{hb} \\ A_{bh} & A_{bb} \end{bmatrix} \begin{bmatrix} U_{h,K} \\ U_{b,K} \end{bmatrix} = \begin{bmatrix} \mathcal{F}_{h,K} \\ \mathcal{F}_{b,K} \end{bmatrix}, \tag{14}$$

where A_{hh} is the local matrix associated to $B_{DG}(u_h^{n+1}, v_h)\Big|_K + \int_K \xi^*(u_h^n)\boldsymbol{\nabla} u_h^{n+1} \cdot \boldsymbol{\nabla} v_h \, d\boldsymbol{x}$; A_{hb} is the local matrix associated to $B_{DG}(u_b^{n+1}, v_h)\Big|_K$; A_{bh} is the local matrix associated to $B_{DG}(u_h^{n+1}, v_b)\Big|_K$; A_{bb} is the local matrix associated to $B_{DG}(u_b^{n+1}, v_b)\Big|_K + \int_K \xi^*(u_h^n)\boldsymbol{\nabla} u_b^{n+1} \cdot \boldsymbol{\nabla} v_b \, d\boldsymbol{x}$; $\mathcal{F}_{h,K}$ is the local vector associated to $F_{DG}(v_h)\Big|_K$, and $\mathcal{F}_{b,K}$ is the local vector associated to $F_{DG}(v_b)\Big|_K$.

As the support of $U_{b,K}$ is contained in the element K, we perform a static condensation of $U_{b,K}$ in eq. (14) to obtain the reduced local problem

$$\mathcal{A}_K U_{h,K} = \mathcal{F}_K,$$

where $\mathcal{A}_K = A_{hh} - A_{hb}(A_{bb})^{-1}A_{bh}$ and $\mathcal{F}_K = \mathcal{F}_{h,K} - A_{hb}(A_{bb})^{-1}\mathcal{F}_{b,K}$. The local matrices and vectors, given by $\mathcal{A}_K$ and $\mathcal{F}_K$ for each element K, are used to obtain the global system that will provide the solution to the problem in eq. (11).

4 Numerical results

In this section we present some numerical experiments in order to evaluate the behavior of the proposed method in the solution of convection-diffusion problems. In addition, these numerical results are compared with those obtained with the DG method, described in eq. (3). The convergence of the nonlinear procedure is attained by setting a tolerance equal to 10^{-7}.

4.1 Problem with parabolic and exponential layers

This problem is a benchmark case, defined over the domain $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$, which exhibits parabolic layers at $y = 0$ and $y = 1$ and an exponential layer at $x = 1$. The coefficients of the equation are: $\kappa = 10^{-4}$, $\boldsymbol{\beta} = (1,0)^T$, $\sigma = 0$, and $f = 1$. Homogeneous Dirichlet boundary conditions are imposed on Γ.

We consider a structured triangulation of $\bar{\Omega}$ with 20 partitions in each direction, x and y, resulting in a mesh with 800 elements. Figure 1 shows the solutions obtained with the DG (left) and BDD (right) methods. The solution obtained by the DG method produces overshoots in the two parabolic layers. These undesirable instabilities disappear in the solution obtained by the proposed formulation, the BDD method, as depicted in the figure on the right.

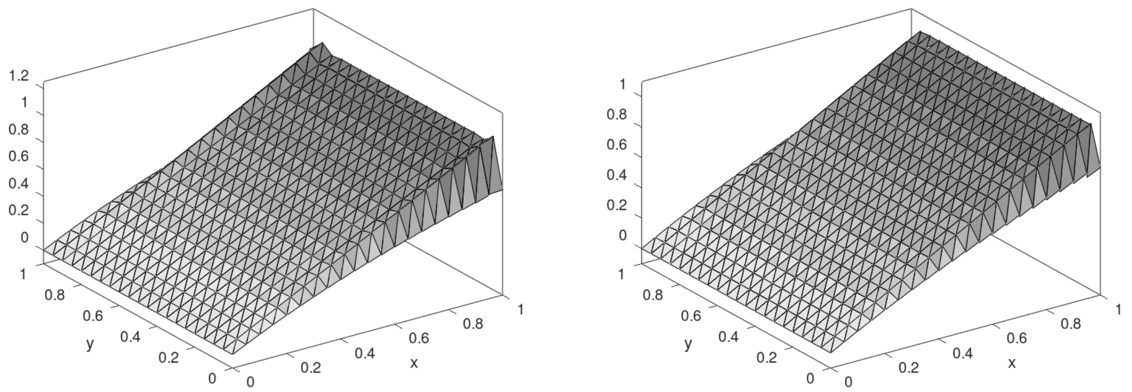

Figure 1. DG (left) and BDD (right) solutions.

4.2 Problem with internal layer

This convection-dominated convective-diffusive problem, defined in the domain $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$, presents a solution with a internal layer. The coefficients of the equation are: $\kappa = 10^{-4}$, $\boldsymbol{\beta} = (1,1)^T$, $\sigma = f = 0$. The Dirichlet boundary conditions are given by

$$u = \begin{cases} 1, & \text{if } x \in [0.5,\ 1],\ y = 0, \text{ or } x = 1,\ y \in [0,\ 0.5]; \\ 0, & \text{otherwise.} \end{cases}$$

Figure 2 displays the solutions obtained with the DG (left) and BDD (right) methods. The DG method solution gives rise to spurious oscillations in the vicinity of the internal boundary layer, whereas the solution obtained by the BDD method is free of numerical instabilities.

4.3 Convergence rates

In this example we evaluate the convergence rates, in the $L^2(\Omega)$ and $H^1(\Omega)$ norms, of the proposed formulation. We consider a convection-diffusion-reaction problem with $\kappa = 10^{-6}$, $\boldsymbol{\beta} = (1,0)^T$ and $\sigma = 1$. The source term, f, and the Dirichlet boundary conditions are given such that the function

$$u(x,y) = sin(\pi x)cos(\pi x)$$

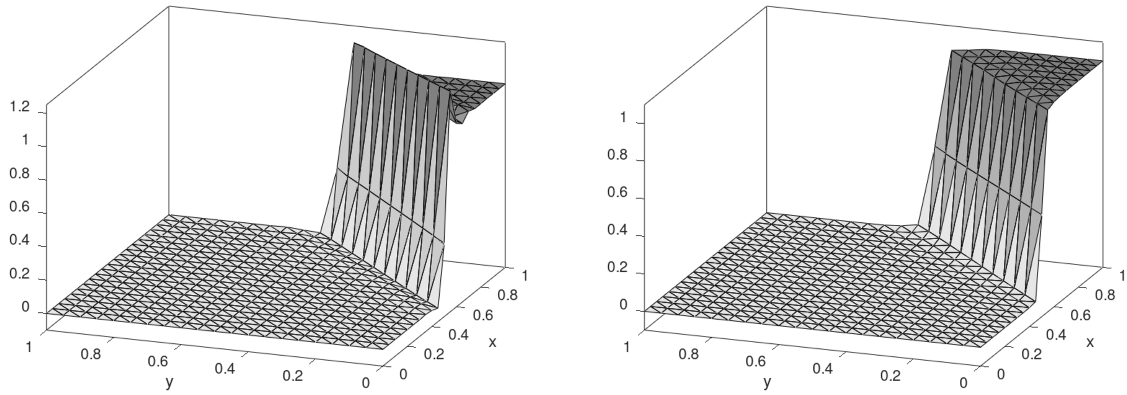

Figure 2. DG (left) and BDD (right) solutions.

is an exact solution to the problem give by eq. (1), in the two-dimensional domain, $\Omega = (0, 1) \times (0, 1)$.

Optimal convergence rates are obtained for the DG and BDD methods, although the highest errors are produced by the BDD method, as shown in Fig. 3.

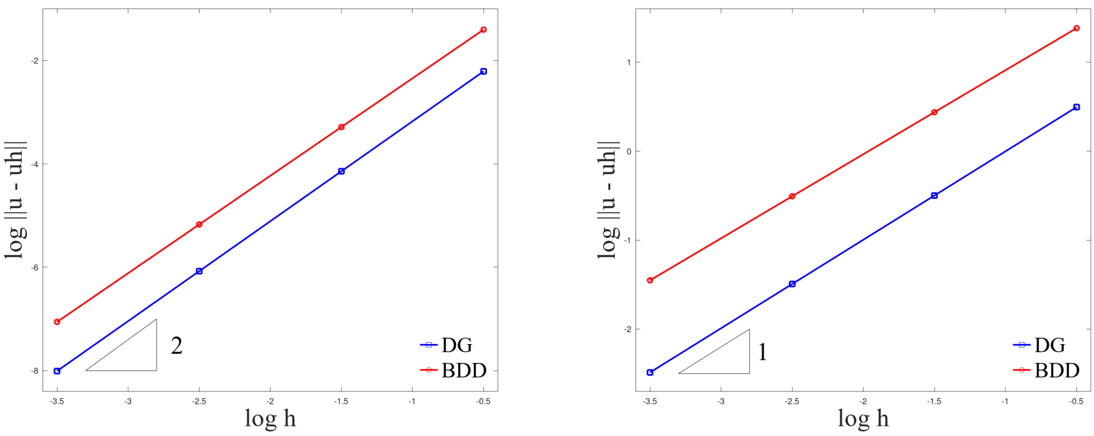

Figure 3. Convergence rates: $L^2(\Omega)$-norm (left) and $H^1(\Omega)$-norm (right).

5 Conclusions

We have presented a discontinuous and nonlinear multiscale method for numerical solution of convection-diffusion-reaction problems. A local and residual-based nonlinear artificial diffusion operator is added to a DG formulation described in a two-scale setting. The proposed method is efficient in eliminating the spurious oscillations that appear near sharp boundary layers, as presented in the numerical experiments. Furthermore, the method presented optimal convergence rates in the $L^1(\Omega)$ and $H^1(\Omega)$ norms.

The developed method offers a promising approach to solving convection-dominated problems. Its ability to effectively represent boundary layers makes it suitable for a wide range of applications in fluid dynamics, heat transfer, and transport processes. Further research and application of this method hold significant potential for enhancing the accuracy and efficiency of numerical simulations in convection-dominated scenarios.

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