Firsts steps for a fully exact thin-walled rod model incorporating generic cross-sectional distortion

Marcos P. Kassab, Eduardo M. B. Campello

Dept. of Structural and Geotechnical Engineering, Polytechnic School, University of São Paulo
P.O. Box 61548, 05424-970, São Paulo, Brazil
marcos.kassab@usp.br, campello@usp.br

Abstract. Thin-walled frame structures consisted of profiles or rod members are often subject to loadings that induce local effects on the profiles´ webs and flanges. For some elements, local instability is the main design constraint. This work aims to present the first steps towards a thin-walled kinematically-exact rod formulation with cross-sectional local effects on its kinematical assumptions. The model´s displacement field allows for the traditional cross-sectional rigid body motion, along with wall´s mid-line in- and out-of-plane deformations (first-order in-plane distortion and out-of-plane warping, respectively, herein called primary deformations) and a kinematically-exact rotation for out-of-mid-line points (herein called secondary deformation). Accordingly, rigid body motion is parametrized as usual for geometrically exact models, whilst primary deformations are described by means of the Generalized Beam Theory (GBT) concept. The (shell-like) rotation that defines the secondary deformation, in turn, is obtained as a function of the walls´ mid-line displacement field and is parametrized using Kirchhoff-Love shell assumptions. This is an ongoing research, and only the kinematical description, the related weak form (for future FEM discretization) are presented. Its linearization is only conceptually briefed by now. Neither constitutive equations nor numerical implementation are addressed here.

Keywords: thin-walled rods, cross-sectional local effects, distortion, warping, finite element method (FEM).

1 Introduction

Usual rod models take into consideration only cross-sectional rigid body displacements, allowing at most an additional warping deformation, for both low-order and kinematically-exact frameworks. Thin-walled rod structures, however, are naturally prone to local deformation effects, which require additional enrichment of the kinematical description. In this context, the Generalized Beam Theory (GBT) was introduced by Schardt [1] in the context of geometric linear theories, describing the body transformation as a linear combination of deformation modes (also called GBT modes) for the thin-wall mid-lines (primary distortion). The GBT approach divides the structural analysis into two steps: 1) a cross-section analysis (to determine the GBT modes); and 2) an element or member analysis (to solve, usually by FEM, for the mode amplitudes). In this approach, the displacements of points located out of the wall´s midlines are obtained by imposition of usual plane-stress assumptions for plates (secondary distortion).

The aforementioned process was devised for application in a linear elastic static context, being later expanded to embed dynamics and second-order buckling analysis. More recently, some developments have been made to incorporate the GBT modes into robust kinematically-exact rod models, as in Gonçalves *et al*. [2] or in Li and Ma [3]. Therein, first insights on how to incorporate local web/plate distortion into such models were provided. Despite achieving satisfactory results, some theoretical limitations were yet to be treated: firstly, the secondary distortion was parametrized in a linearized fashion. Secondly, either the constitutive equation or the strain measures were truncated, leading to a lower order constitutive equation.

This work seeks to contribute to overcome those limitations. The proposed approach is to, given a set of distortional modes for the primary distortion (obtained through the GBT concept), parametrize the secondary deformation using expressions for the rotation from the kinematically-exact Kirchhoff-Love shell theory. We also intend to allow for the use of more consistent (non-truncated) material laws. For the current formulation, the following notation is used: lowercase Latin or Greek letters $(a, b, \dots, \alpha, \beta, \dots.)$ denote scalar quantities, bold lowercase Latin or Greek letters $(\boldsymbol{a}, \boldsymbol{b}, \dots, \boldsymbol{\alpha}, \boldsymbol{\beta}, \dots.)$ denote vectors, Bold capital Latin or Greek letters $(\boldsymbol{A}, \boldsymbol{B}, \dots)$ denote second-order tensors. Implicit summation convention is used throughout.

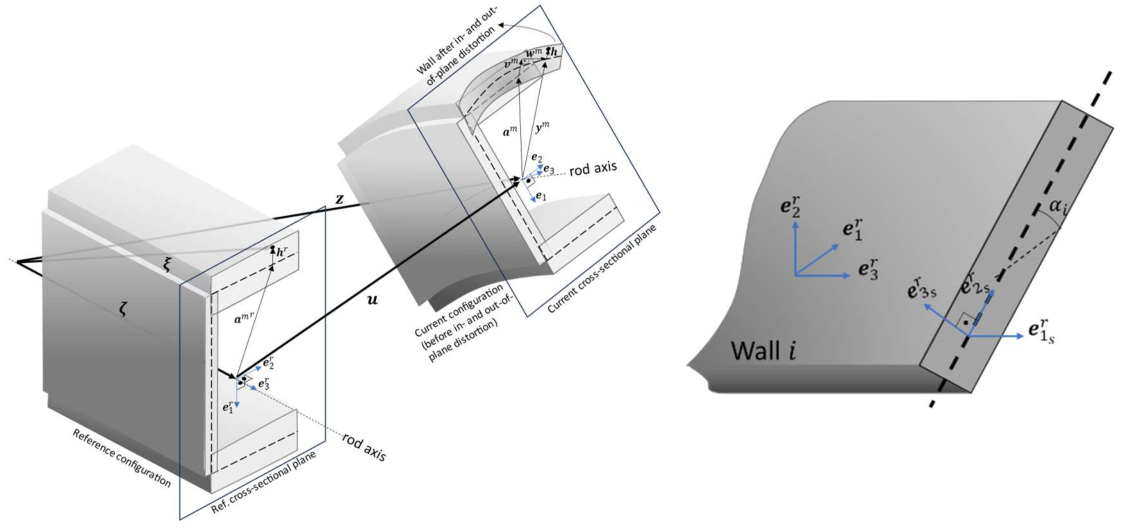
When indices are Greek letters, they range from 1 to 2, and when they are Latin letters, from 1 to 3. Scalar, cross and dyadic products are represented by "·", "×" and "⊗", respectively. The comma in $(\circ)_{,g}$ indicates partial derivative w.r.t. g.

2 The proposed rod model

The proposed rod model is intended to be suitable for prismatic rods (i.e., with constant cross-section) and straight initial configuration. The cross-sections are considered to be composed of thin-walled rectangular walls, and arbitrary (closed, open, branched, different thickness for each wall, etc.) geometries.

2.1 Kinematical description

The kinematical description is divided into two parts: the primary deformation (for walls´ mid-line points) and the secondary deformation (for the remaining, out-of-mid-line points). The primary deformation is function of **a)** the six usual rigid body degrees of freedom (three for the axial displacements $\boldsymbol{u}$ and three for the cross-sectional rotations $\boldsymbol{\theta}$, the latter being calculated by the Euler-Rodrigues formula as in Yojo, Pimenta and Campello [4]–[6]) and **b)** a set of n_v in-plane (ϕ_i) and n_w out-of plane (ψ_j) arbitrary deformation modes. Those deformation modes can be interpreted as a basis and, in the current work, a detailed discussion about how to generate them is out of the scope – it suffices to say that they will be derived through the GBT cross-sectional analysis, analogously as in [7], [8]. It is important to define now two reference systems: one for the cross-section $\{\boldsymbol{e}_1^r, \boldsymbol{e}_2^r, \boldsymbol{e}_3^r\}$, with e_3^r aligned with the reference rod axis and one for each individual wall $\{\boldsymbol{e}_{1_s}^r, \boldsymbol{e}_{2_s}^r, \boldsymbol{e}_{3_s}^r\}$, with $\boldsymbol{e}_{1_s}^r$ aligned with $\boldsymbol{e}_3^r$, $\boldsymbol{e}_{2_s}^r$ aligned with the wall mid-line and $\boldsymbol{e}_{3_s}^r$ orthogonal to the two latter (see fig. 1b).



a) b)

Figure 1 – Cross-section a) deformation b) wall geometry c) detail of mid-line deformation

As can be seen in Fig. 1a), points in the reference configuration are given by

$$\boldsymbol{\xi} = \boldsymbol{\zeta} + \boldsymbol{a}^r, \tag{1}$$

$$\boldsymbol{a}^r = \boldsymbol{a}^{m^r} + \boldsymbol{h}^r, \tag{2}$$

where $\boldsymbol{\zeta} = \zeta \boldsymbol{e}_3^r$ (with $\zeta \in \Omega^r = [0, L]$) is the axis reference position and $\boldsymbol{a}^r$ is the position of an arbitrary point of the cross-section, that can be decomposed in two parts: a projection in the mid-line of the wall that it belongs to ($\boldsymbol{a}^{m^r}$) and a position (or wall director) $\boldsymbol{h}^r$ relative to this latter that is parallel to the wall thickness.

Points in the current configuration are given by

$$\boldsymbol{x} = \boldsymbol{x}^m + \boldsymbol{x}^h = \boldsymbol{z} + \boldsymbol{y}^m + \boldsymbol{h}, \tag{3}$$

where $\boldsymbol{x}^m = \boldsymbol{z} + \boldsymbol{y}^m$ is the current position of the wall´s mid-line points (with $\boldsymbol{z}$ as the current position of the axis), $\boldsymbol{y}^m = \boldsymbol{a}^m + \boldsymbol{v}^m + \boldsymbol{w}^m$ is the current position of the mid-line points w.r.t. the axis (i.e., the primary deformation) and $\boldsymbol{x}^h = \boldsymbol{h}$ is the final position (or director) of the other points w.r.t the mid-line (i.e. the secondary deformation). Let us detail each of the components of the displacement field: the axis current position is determined by the axial displacement $\boldsymbol{u}$ through

$$\boldsymbol{z} = \boldsymbol{\zeta} + \boldsymbol{u}. \tag{4}$$

The wall midline rotation w.r.t. the axis is

$$\boldsymbol{a}^m = \boldsymbol{Q}\boldsymbol{a}^{m^r}, \tag{5}$$

where $\boldsymbol{Q}$ is the rotation tensor of the cross-section. Note that $\boldsymbol{e}_3 = \boldsymbol{Q}\boldsymbol{e}_3^r$ is not tangent to the rod axis, in general.

The cross-sectional in- and out-of plane distortions of the mid-line (the out-of-plane standing for the warping) are, respectively

$$\boldsymbol{v}^m = (\boldsymbol{\phi}_\beta \cdot \boldsymbol{r})\boldsymbol{e}_\beta \text{ and } \boldsymbol{w}^m = (\boldsymbol{\psi} \cdot \boldsymbol{p})\boldsymbol{e}_3, \tag{6}$$

where $\boldsymbol{\phi}_\beta$ and $\boldsymbol{\psi}$ are vectors that collect the in- and out-of-plane distortion modes of the mid-line and $\boldsymbol{r}$ and $\boldsymbol{p}$ are the intensities of the associated modes. Lastly, once the primary deformation is established, the secondary deformation can be obtained as a function of the primary distortion, by means of the Kirchhoff-Love shell assumption. Thus, it is merely a rotation, characterized by a tensor $\boldsymbol{Q}^h$, of the director $\boldsymbol{h}^r$, such that

$$\boldsymbol{h} = \boldsymbol{Q}^h\boldsymbol{h}^r. \tag{7}$$

This implies that the wall director remains perpendicular to the wall´s mid-line after the deformation.

It is possible to express the deformation gradient by its column-vectors and separate all the terms that contain $\boldsymbol{h}^r$ from the rest. The terms that do not contain it are grouped and named here *mid-line* terms ($\boldsymbol{f}_i^m$) (some GBT-related authors call them *membrane* terms), and the ones that contain it are grouped and named *thickness* terms ($\boldsymbol{f}_i^h$). Putting the cross-sectional rotation tensor in evidence, it is possible to write

$$\boldsymbol{F} = \boldsymbol{x}_{,i} \otimes \boldsymbol{e}_i^r = \boldsymbol{Q}\boldsymbol{F}^r = \boldsymbol{Q}\left(\boldsymbol{f}_i^{m^r} + \boldsymbol{f}_i^{h^r}\right) \otimes \boldsymbol{e}_i^r, \tag{8}$$

where $(\circ)^r$ are defined as back-rotated quantities. Using eq. (3)-(8), one gets

$$\boldsymbol{f}_\alpha^{m^r} = l_\alpha^2\boldsymbol{e}_\alpha^r + \left(\boldsymbol{\phi}_{\beta,\alpha} \cdot \boldsymbol{r}\right)\boldsymbol{e}_\beta^r + \left(\boldsymbol{\psi}_{,\alpha} \cdot \boldsymbol{p}\right)\boldsymbol{e}_3^r, \tag{9}$$

$$\boldsymbol{f}_3^{m^r} = \boldsymbol{\eta}^r + \boldsymbol{e}_3^r + \boldsymbol{\kappa}^r \times \boldsymbol{y}^{m^r} + \left(\boldsymbol{\phi}_\beta \cdot \boldsymbol{r}'\right)\boldsymbol{e}_\beta^r + (\boldsymbol{\psi} \cdot \boldsymbol{p}')\boldsymbol{e}_3^r, \tag{10}$$

$$\boldsymbol{f}_\alpha^{h^r} = \left(h^r\boldsymbol{Q}^T\boldsymbol{Q}_{,\alpha}^h - \epsilon_{\alpha\beta}l_\beta\boldsymbol{Q}^{h^r}\right)\left(-\epsilon_{\gamma\beta}l_\beta\right)\boldsymbol{e}_\gamma^r, \tag{11}$$

$$\boldsymbol{f}_3^{h^r} = \boldsymbol{Q}^T\boldsymbol{Q}^{h'}\boldsymbol{h}^r, \tag{12}$$

where $\boldsymbol{\eta}^r = \boldsymbol{Q}^T\boldsymbol{\eta} = \boldsymbol{Q}^T(\boldsymbol{u}' + \boldsymbol{e}_3^r - \boldsymbol{e}_3)$, $\boldsymbol{\kappa}^r = \boldsymbol{Q}^T\boldsymbol{\kappa}$ (with $\boldsymbol{\kappa} = axial(\boldsymbol{Q}'\boldsymbol{Q}^T)$ as defined in Campello and Pimenta [9]), $\boldsymbol{y}^{m^r} = \boldsymbol{Q}^T\boldsymbol{y}^m$, $h^r = \|\boldsymbol{h}^r\|$, $l_1 = \cos(\alpha_i)$, $l_2 = \sin(\alpha_i)$ (α_i is the inclination of the wall to which the analysed point belongs (w.r.t. direction $\boldsymbol{e}_1^r$ of the reference system, see Fig. 1b), and $\epsilon_{\alpha\beta} = \left(\boldsymbol{e}_\alpha^r \times \boldsymbol{e}_\beta^r\right) \cdot \boldsymbol{e}_3^r$.

It is now convenient to define the explicit expression for the director´s Kirchhoff-Love shell rotation $\boldsymbol{Q}^h$. From Pimenta, Neto and Campello [10], expressing in terms of the cross-section local system instead of a shell local system, one gets

$$\boldsymbol{Q}^h = \boldsymbol{Q}\left\{\left\|\epsilon_{\alpha\beta}l_\alpha\boldsymbol{g}_\beta^{m^r}\right\|^{-1}\left(\frac{\epsilon_{\alpha\beta}l_\alpha^2}{\left\|\boldsymbol{f}_3^{m^r}\right\|}\boldsymbol{g}_\beta^{m^r} \times \boldsymbol{f}_3^{m^r} - \epsilon_{\alpha\beta}l_\alpha l_\beta\boldsymbol{g}_\beta^{m^r}\right) \otimes \boldsymbol{e}_\alpha^r + \left\|\boldsymbol{f}_3^{m^r}\right\|^{-1}\boldsymbol{f}_3^{m^r} \otimes \boldsymbol{e}_3^r\right\}. \tag{13}$$

Note that $\boldsymbol{Q}^h$ can be re-written as $\boldsymbol{Q}^h = \boldsymbol{Q}\boldsymbol{Q}^{h^r}$, where $\boldsymbol{Q}^{h^r}$ is defined as the back-rotated shell rotation.

Now, some useful derivatives are shown below:

$$\boldsymbol{Q}_{,1_s}^h = \boldsymbol{K}_{1s}\boldsymbol{Q}^h \text{ and } \left(\boldsymbol{Q}^{h^r}\right)_{,1_s} = \boldsymbol{Q}^T\left(\boldsymbol{Q}^{h'} - \boldsymbol{K}\boldsymbol{Q}^h\right), \tag{14}$$

$$\boldsymbol{Q}_{,2_s}^h = \boldsymbol{K}_{2_s}\boldsymbol{Q}^h \text{ and } \left(\boldsymbol{Q}^{h^r}\right)_{,2_s} = \boldsymbol{Q}^T\boldsymbol{K}_{2_s}\boldsymbol{Q}^{h^r}, \tag{15}$$

$$\left(\boldsymbol{Q}^{h^r}\right)_{,3_s} = \boldsymbol{0}, \tag{16}$$

and

$$Q^h_{,\alpha} = l_\alpha K_{2_s} Q^h. \tag{17}$$

with the shell curvature vectors being defined as

$$\kappa_{\alpha_s} = axial(K_{\alpha_s}) = \Gamma_{\beta_s} u^m_{,\beta_s\alpha_s}, \tag{18}$$

where Γ_{β_s} is defined analogously as in [10] and

$$u^m = x^m - (\zeta + a^{mr}). \tag{19}$$

Using eqs. (14)-(18), it is possible to rewrite eq. (11) and (12) as

$$f^{h^r}_\alpha = (l_\alpha h^r Q^T K_{2_s} Q - \epsilon_{\alpha\beta} l_\beta I)(-\epsilon_{\gamma\beta} l_\beta) Q^{h^r} e^r_\gamma, \tag{20}$$

$$f^{h^r}_3 = Q^T K_{1_s} Q Q^{h^r} h^r, \tag{21}$$

Now, the second-order derivatives $u^m_{,\beta_s\alpha_s}$ are needed. Performing the algebra, one gets

$$u^m_{,1_s} = Q f^{m^r}_3 + e^r_3 \text{ and } u^m_{,2_s} = Q l_\alpha f^{m^r}_\alpha, \tag{22}$$

$$u^m_{,1_s1_s} = KQ f^{m^r}_3 + Q f^{m^{r\prime}}_3, \tag{23}$$

$$u^m_{,1_s2_s} = Q l_\alpha f^{m\ r}_{3,\alpha} = u^m_{,2_s1_s} = KQ l_\alpha f^{m^r}_\alpha + Q l_\alpha f^{m^{r\prime}}_\alpha, \tag{24}$$

$$u^m_{,2_s2_s} = Q l_\alpha l_\beta f^{m\ r}_{\alpha,\beta}, \tag{25}$$

where the following additional auxiliary results appear:

$$f^m_{\alpha,\gamma} = (\phi_{\beta,\alpha\gamma} \cdot r) e^r_\beta + (\psi_{,\alpha\gamma} \cdot p) e^r_3, \tag{26}$$

$$f^{m\prime}_\alpha = (\phi_{\beta,\alpha} \cdot r') e^r_\beta + (\psi_{,\alpha} \cdot p') e^r_3, \tag{27}$$

$$f^m_{3,\gamma} = \kappa^r \times [(\phi_{\beta,\alpha} \cdot r) e^r_\beta + (\psi_{,\alpha\gamma} \cdot p) e^r_3] + (\phi_{\beta,\alpha} \cdot r') e^r_\beta + (\psi_{,\alpha} \cdot p') e^r_3 \tag{28}$$

$$f^{m\prime}_3 = \eta^r - y^{m^r} \times \kappa^r + \kappa^r \times [(\phi_\beta \cdot r') e^r_\beta + (\psi \cdot p') e^r_3] + (\phi_\beta \cdot r'') e^r_\beta + (\psi \cdot p'') e^r_3 \tag{29}$$

The displacement field x and the deformation gradient F are now fully characterized.

2.2 Equilibrium: virtual work

The Virtual Work Theorem is applied to the rod. As in previous works, (for example Yojo and Pimenta [4] or Campello and Pimenta [5]), the internal virtual work can be expressed as function of the back-rotated first Piola-Kirchhoff stress tensor $P^r = Q^T P = \tau^r_i \otimes e^r_i$ and the back-rotated deformation gradient $F^r = Q^T F$. Accordingly, one gets

$$\delta W = \delta W_{int} - \delta W_{ext} = 0, \tag{30}$$

with

$$\delta W_{int} = \int_{\Omega^r}\int_{A^r} P : \delta F dA^r d\Omega^r = \int_{\Omega^r}\int_{A^r} P^r : \delta F^r dA^r d\Omega^r = \int_{\Omega^r}\int_{A^r} \tau^r_i \cdot \delta f^r_i dA^r d\Omega^r =$$

$$= \int_{\Omega^r}\int_{A^r} (\tau^r_i \cdot \delta f^{m^r}_i + \tau^r_i \cdot \delta f^{h^r}_i) dA^r d\Omega^r = \delta W^m_{int} + \delta W^h_{int} \tag{31}$$

$$\delta W_{ext} = \int_{\Omega^r}\int_{A^r} \bar{b} \cdot \delta x dA^r d\Omega^r + \int_{C^r}\int_{A^r} \bar{t} \cdot \delta x dA^r dC^r = \delta W^b_{ext} + \delta W^t_{ext}, \tag{32}$$

where δW^m_{int} are the mid-line terms of the internal virtual work and δW^h_{int} are the thickness terms. Note that the external virtual work has terms from both surface and body external loadings (δW^b_{ext} and δW^t_{ext}). Expanding the area integrals from eq. (31) one gets, for the mid-line internal work,

$$\int_{A^r} (\tau^r_i \cdot \delta f^{m^r}_i) dA^r = \sigma^{m^r}_1 \cdot \delta \varepsilon^r_1, \tag{33}$$

with

$$\sigma^{m^r}_1 = [n^{m^r} \quad m^{m^r} \quad \rho^m \quad \pi^m \quad \beta^m \quad \alpha^m]^T, \tag{34}$$

$$\delta \varepsilon^r_1 = [\delta \eta^r \quad \delta \kappa^r \quad \delta r \quad \delta p \quad \delta r' \quad \delta p']^T, \tag{35}$$

$$\boldsymbol{n}^{m^r} = \int_{A^r}(\boldsymbol{\tau}_3^r)dA^r, \tag{36}$$

$$\boldsymbol{m}^{m^r} = \int_{A^r}(\boldsymbol{y}^{m^r} \times \boldsymbol{\tau}_3^r)dA^r\ , \tag{37}$$

$$\boldsymbol{\rho}^m = \int_{A^r}\big([\boldsymbol{\phi}_\beta \otimes (\boldsymbol{\kappa}^r \times \boldsymbol{e}_\beta^r)]\boldsymbol{\tau}_3^r + (\boldsymbol{\phi}_{\beta,\alpha} \otimes \boldsymbol{e}_\beta^r)\boldsymbol{\tau}_\alpha^r\big)dA^r, \tag{38}$$

$$\boldsymbol{\pi}^m = \int_{A^r}\big([\boldsymbol{\psi} \otimes (\boldsymbol{\kappa}^r \times \boldsymbol{e}_3^r)]\boldsymbol{\tau}_3^r + (\boldsymbol{\psi}_{,\alpha} \otimes \boldsymbol{e}_3^r)\boldsymbol{\tau}_\alpha^r\big)dA^r, \tag{39}$$

$$\boldsymbol{\beta}^m = \int_{A^r}\left((\boldsymbol{\phi}_\beta \otimes \boldsymbol{e}_\beta^r)\boldsymbol{\tau}_3^r\right)dA^r, \tag{40}$$

$$\boldsymbol{\alpha}^m = \int_{A^r}\big((\boldsymbol{\psi} \otimes \boldsymbol{e}_3^r)\boldsymbol{\tau}_3^r\big)dA^r, \tag{41}$$

and for the thickness internal work

$$\int_{A^r}\big(\boldsymbol{\tau}_i^r \cdot \delta \boldsymbol{f}_i^{h^r}\big)dA^r = \boldsymbol{\sigma}_1^{h^r} \cdot \delta\boldsymbol{\varepsilon}_1^r + \boldsymbol{\sigma}_2^{h^r} \cdot \delta\boldsymbol{\varepsilon}_2^r \tag{42}$$

with

$$\boldsymbol{\sigma}_1^{h^r} = \left[\boldsymbol{n}^{h^r} \quad \boldsymbol{m}^{h^r} \quad \boldsymbol{\rho}^h \quad \boldsymbol{\pi}^h \quad \boldsymbol{\beta}^h \quad \boldsymbol{\alpha}^h\right]^T, \tag{43}$$

$$\boldsymbol{\sigma}_2^{h^r} = \left[\boldsymbol{n}_\theta^{h^r} \quad \boldsymbol{p}^{h^r} \quad \boldsymbol{q}^{h^r} \quad \boldsymbol{\delta}^h \quad \boldsymbol{\chi}^h\right]^T, \tag{44}$$

$$\delta\boldsymbol{\varepsilon}_2^r = [\delta\boldsymbol{\theta} \quad \delta\boldsymbol{\eta}^{r\prime} \quad \delta\boldsymbol{\kappa}^{r\prime} \quad \delta\boldsymbol{r}'' \quad \delta\boldsymbol{p}'']^T, \tag{45}$$

$$\boldsymbol{n}^{h^r} = \int_{A^r}\big(\boldsymbol{t}_{\alpha f_3} + \boldsymbol{t}_{3f_3}\big)dA^r, \tag{46}$$

$$\boldsymbol{m}^{h^r} = \int_{A^r}\left(\boldsymbol{t}_{3\kappa} + \boldsymbol{y}^{m^r} \times \big(\boldsymbol{t}_{\alpha f_3} + \boldsymbol{t}_{3f_3}\big) + \boldsymbol{y}_{,\alpha}^{m^r} \times \boldsymbol{t}_{\alpha f_{3,\gamma}} + \boldsymbol{y}^{m^r\prime} \times \boldsymbol{t}_{3f_3'}\right)dA^r, \tag{47}$$

$$\begin{aligned}\boldsymbol{\rho}^h = \int_{A^r}\Big(&(\boldsymbol{\phi}_{\gamma,\beta} \otimes \boldsymbol{e}_\gamma^r)(\boldsymbol{t}_{\alpha f_\beta} + \boldsymbol{t}_{3f_\beta}) - (\boldsymbol{\phi}_\beta \otimes \boldsymbol{e}_\beta^r)\boldsymbol{K}^r\big(\boldsymbol{t}_{\alpha f_3} + \boldsymbol{t}_{3f_3}\big) + (\boldsymbol{\phi}_{\delta,\beta\gamma} \otimes \boldsymbol{e}_\delta^r)\boldsymbol{t}_{\alpha f_{\beta,\gamma}} - \\ &-(\boldsymbol{\phi}_{\beta,\gamma} \otimes \boldsymbol{e}_\beta^r)\boldsymbol{K}^r\boldsymbol{t}_{\alpha f_{3,\gamma}} - (\boldsymbol{\phi}_\beta \otimes \boldsymbol{e}_\beta^r)\boldsymbol{K}^{r\prime}\boldsymbol{t}_{3f_3'}\Big)dA^r\ ,\end{aligned} \tag{48}$$

$$\begin{aligned}\boldsymbol{\pi}^h = \int_{A^r}\Big(&(\boldsymbol{\psi}_{,\beta} \otimes \boldsymbol{e}_3^r)(\boldsymbol{t}_{\alpha f_\beta} + \boldsymbol{t}_{3f_\beta}) - (\boldsymbol{\psi} \otimes \boldsymbol{e}_3^r)\boldsymbol{K}^r\big(\boldsymbol{t}_{\alpha f_3} + \boldsymbol{t}_{3f_3}\big) + (\boldsymbol{\psi}_{,\beta\gamma} \otimes \boldsymbol{e}_3^r)\boldsymbol{t}_{\alpha f_{\beta,\gamma}} - \\ &-(\boldsymbol{\psi}_{,\gamma} \otimes \boldsymbol{e}_3^r)\boldsymbol{K}^r\boldsymbol{t}_{\alpha f_{3,\gamma}} - (\boldsymbol{\psi} \otimes \boldsymbol{e}_3^r)\boldsymbol{K}^{r\prime}\boldsymbol{t}_{3f_3'}\Big)dA^r,\end{aligned} \tag{49}$$

$$\boldsymbol{\beta}^h = \int_{A^r}\Big((\boldsymbol{\phi}_\beta \otimes \boldsymbol{e}_\beta^r)\big(\boldsymbol{t}_{\alpha f_3} + \boldsymbol{t}_{3f_3}\big) + (\boldsymbol{\phi}_{\beta,\gamma} \otimes \boldsymbol{e}_\beta^r)\left(\boldsymbol{t}_{\alpha f_{3,\gamma}} + \boldsymbol{t}_{3f_{\gamma'}}\right) - (\boldsymbol{\phi}_\beta \otimes \boldsymbol{e}_\beta^r)\boldsymbol{K}^r\boldsymbol{t}_{3f_3'}\Big)dA^r, \tag{50}$$

$$\boldsymbol{\alpha}^h = \int_{A^r}\Big((\boldsymbol{\psi} \otimes \boldsymbol{e}_3^r)\big(\boldsymbol{t}_{\alpha f_3} + \boldsymbol{t}_{3f_3}\big) + (\boldsymbol{\psi}_{,\gamma} \otimes \boldsymbol{e}_3^r)\left(\boldsymbol{t}_{\alpha f_{3,\gamma}} + \boldsymbol{t}_{3f_{\gamma'}}\right) - (\boldsymbol{\psi} \otimes \boldsymbol{e}_3^r)\boldsymbol{K}^r\boldsymbol{t}_{3f_3'}\Big)dA^r, \tag{51}$$

$$\boldsymbol{n}_\theta^h = \int_{A^r}(\boldsymbol{t}_{\alpha\theta} + \boldsymbol{t}_{3\theta})dA^r, \tag{52}$$

$$\boldsymbol{p}^{h^r} = \int_{A^r}\left(\boldsymbol{t}_{3f_3'}\right)dA^r, \tag{53}$$

$$\boldsymbol{q}^{h^r} = \int_{A^r}\left(\boldsymbol{y}^{m^r} \times \boldsymbol{t}_{3f_3'}\right)dA^r, \tag{54}$$

$$\boldsymbol{\delta}^h = \int_{A^r}\left((\boldsymbol{\phi}_\beta \otimes \boldsymbol{e}_\beta^r)\boldsymbol{t}_{3f_3'}\right)dA^r, \tag{55}$$

$$\boldsymbol{\chi}^h = \int_{A^r}\left((\boldsymbol{\psi} \otimes \boldsymbol{e}_3^r)\boldsymbol{t}_{3f_3'}\right)dA^r, \tag{56}$$

with $\boldsymbol{t}_{iv}$ being the vectors that are conjugated with a given vector $\delta\boldsymbol{v}$ w.r.t $\boldsymbol{\tau}_i^r$, when the operation $\boldsymbol{\tau}_i^r \cdot \delta\boldsymbol{f}_i^{h^r}$ is performed. The complete expression for each $\boldsymbol{t}_{iv}$ has an intricate form and the authors have chosen to omit it for conciseness. For the external work, one gets

$$\int_{A^r}\bar{\boldsymbol{b}} \cdot \delta\boldsymbol{x}dA^r = \left(\bar{\boldsymbol{q}}_1{}^m + \bar{\boldsymbol{q}}_1{}^h\right) \cdot \delta\boldsymbol{d}_1 + \bar{\boldsymbol{q}}_2{}^h \cdot \delta\boldsymbol{d}_2, \tag{57}$$

where

$$\bar{\boldsymbol{q}}_1{}^m = [\bar{\boldsymbol{n}}^m \quad \boldsymbol{\Gamma}^T\bar{\boldsymbol{m}}^m \quad \bar{\boldsymbol{\beta}}^m \quad \bar{\boldsymbol{\alpha}}^m]^T \text{ and } \bar{\boldsymbol{q}}_1{}^h = [\bar{\boldsymbol{n}}^h \quad \boldsymbol{\Gamma}^T\bar{\boldsymbol{m}}^h \quad \bar{\boldsymbol{\beta}}^h \quad \bar{\boldsymbol{\alpha}}^h]^T, \tag{58}$$

$$\bar{\boldsymbol{q}}_2{}^h = [\bar{\boldsymbol{p}}^h \quad \bar{\boldsymbol{q}}^h \quad \bar{\boldsymbol{\rho}}^h \quad \bar{\boldsymbol{\pi}}^h]^T, \tag{59}$$

$$\delta\boldsymbol{d}_1 = [\delta\boldsymbol{u} \quad \delta\boldsymbol{\theta} \quad \delta\boldsymbol{r} \quad \delta\boldsymbol{p}]^T \text{ and } \delta\boldsymbol{d}_2 = [\delta\boldsymbol{\eta}^r \quad \delta\boldsymbol{\kappa}^r \quad \delta\boldsymbol{r}' \quad \delta\boldsymbol{p}']^T. \tag{60}$$

with mid-line contributions

$$\bar{\boldsymbol{n}}^m = \bar{\boldsymbol{n}}^{b,m} + \bar{\boldsymbol{n}}^{t,m} = \int_{A^r} \bar{\boldsymbol{b}}\, dA^r + \int_{C^r} \bar{\boldsymbol{t}}\, dC^r, \quad (61)$$

$$\bar{\boldsymbol{m}}^m = \bar{\boldsymbol{m}}^{b,m} + \bar{\boldsymbol{m}}^{t,m} = \int_{A^r} (\boldsymbol{y}^m \times \bar{\boldsymbol{b}})\, dA^r + \int_{C^r} (\boldsymbol{y}^m \times \bar{\boldsymbol{t}})\, dC^r, \quad (62)$$

$$\bar{\boldsymbol{\beta}}^m = \bar{\boldsymbol{\beta}}^{b,m} + \bar{\boldsymbol{\beta}}^{t,m} = \int_{A^r} (\boldsymbol{\phi}_\beta \otimes \boldsymbol{e}_\beta)\bar{\boldsymbol{b}}\, dA^r + \int_{C^r} (\boldsymbol{\phi}_\beta \otimes \boldsymbol{e}_\beta)\bar{\boldsymbol{t}}\, dC^r, \quad (63)$$

$$\bar{\boldsymbol{\alpha}}^m = \bar{\boldsymbol{\alpha}}^{b,m} + \bar{\boldsymbol{\alpha}}^{t,m} = \int_{A^r} (\boldsymbol{\psi} \otimes \boldsymbol{e}_3)\bar{\boldsymbol{b}}\, dA^r + \int_{C^r} (\boldsymbol{\psi} \otimes \boldsymbol{e}_3)\bar{\boldsymbol{t}}\, dC^r, \quad (64)$$

and thickness contributions

$$\bar{\boldsymbol{n}}^h = \bar{\boldsymbol{n}}^{b,h} + \bar{\boldsymbol{n}}^{t,h} = \boldsymbol{o}, \quad (65)$$

$$\bar{\boldsymbol{m}}^h = \bar{\boldsymbol{m}}^{b,h} + \bar{\boldsymbol{m}}^{t,h} = \int_{A^r} \boldsymbol{\Gamma}^T \left(\boldsymbol{F}_3^m \boldsymbol{\Gamma}_{1_s}^T + l_\alpha \boldsymbol{F}_\alpha^m \boldsymbol{\Gamma}_{2_s}^T\right)(\boldsymbol{h} \times \bar{\boldsymbol{b}})\, dA^r + \\ + \int_{C^r} \boldsymbol{\Gamma}^T \left(\boldsymbol{F}_3^m \boldsymbol{\Gamma}_{1_s}^T + l_\alpha \boldsymbol{F}_\alpha^m \boldsymbol{\Gamma}_{2_s}^T\right)(\boldsymbol{h} \times \bar{\boldsymbol{t}})\, dC^r, \quad (66)$$

$$\bar{\boldsymbol{\beta}}^h = \bar{\boldsymbol{\beta}}^{b,h} + \bar{\boldsymbol{\beta}}^{t,h} = \int_{A^r} \left[-(\boldsymbol{\phi}_\beta \otimes \boldsymbol{e}_\beta^r)\boldsymbol{K}^r \boldsymbol{Q}^T \boldsymbol{\Gamma}_{1_s}^T + (\boldsymbol{\phi}_{\beta,\alpha} \otimes \boldsymbol{e}_\beta^r) l_\alpha \boldsymbol{Q}^T \boldsymbol{\Gamma}_{2_s}^T\right](\boldsymbol{h} \times \bar{\boldsymbol{b}})\, dA^r + \\ + \int_{C^r} \left[-(\boldsymbol{\phi}_\beta \otimes \boldsymbol{e}_\beta^r)\boldsymbol{K}^r \boldsymbol{Q}^T \boldsymbol{\Gamma}_{1_s}^T + (\boldsymbol{\phi}_{\beta,\alpha} \otimes \boldsymbol{e}_\beta^r) l_\alpha \boldsymbol{Q}^T \boldsymbol{\Gamma}_{2_s}^T\right](\boldsymbol{h} \times \bar{\boldsymbol{t}}) dC^r, \quad (67)$$

$$\bar{\boldsymbol{\alpha}}^h = \bar{\boldsymbol{\alpha}}^{b,h} + \bar{\boldsymbol{\alpha}}^{t,h} = \int_{A^r} \left[-(\boldsymbol{\psi} \otimes \boldsymbol{e}_3^r)\boldsymbol{K}^r \boldsymbol{Q}^T \boldsymbol{\Gamma}_{1_s}^T + (\boldsymbol{\psi}_{,\alpha} \otimes \boldsymbol{e}_3^r) l_\alpha \boldsymbol{Q}^T \boldsymbol{\Gamma}_{2_s}^T\right](\boldsymbol{h} \times \bar{\boldsymbol{b}})\, dA^r + \\ + \int_{C^r} \left[-(\boldsymbol{\psi} \otimes \boldsymbol{e}_3^r)\boldsymbol{K}^r \boldsymbol{Q}^T \boldsymbol{\Gamma}_{1_s}^T + (\boldsymbol{\psi}_{,\alpha} \otimes \boldsymbol{e}_3^r) l_\alpha \boldsymbol{Q}^T \boldsymbol{\Gamma}_{2_s}^T\right](\boldsymbol{h} \times \bar{\boldsymbol{t}})\, dC^r, \quad (68)$$

$$\bar{\boldsymbol{p}}^h = \bar{\boldsymbol{p}}^{b,h} + \bar{\boldsymbol{p}}^{t,h} = \int_{A^r} \boldsymbol{Q}^T \boldsymbol{\Gamma}_{1_s}^T (\boldsymbol{h} \times \bar{\boldsymbol{b}}) dA^r + \int_{C^r} \boldsymbol{Q}^T \boldsymbol{\Gamma}_{1_s}^T (\boldsymbol{h} \times \bar{\boldsymbol{t}}) dC^r\,, \quad (69)$$

$$\bar{\boldsymbol{q}}^h = \bar{\boldsymbol{q}}^{b,h} + \bar{\boldsymbol{q}}^{t,h} = \int_{A^r} \boldsymbol{y}^{m^r} \times \left(\boldsymbol{Q}^T \boldsymbol{\Gamma}_{1_s}^T (\boldsymbol{h} \times \bar{\boldsymbol{b}})\right) dA^r + \int_{C^r} \boldsymbol{y}^{m^r} \times \left(\boldsymbol{Q}^T \boldsymbol{\Gamma}_{1_s}^T (\boldsymbol{h} \times \bar{\boldsymbol{t}})\right) dC^r, \quad (70)$$

$$\bar{\boldsymbol{\rho}}^h = \bar{\boldsymbol{\rho}}^{b,h} + \bar{\boldsymbol{\rho}}^{t,h} = \int_{A^r} (\boldsymbol{\phi}_\beta \otimes \boldsymbol{e}_\beta^r) \boldsymbol{Q}^T \boldsymbol{\Gamma}_{1_s}^T (\boldsymbol{h} \times \bar{\boldsymbol{b}}) dA^r + \int_{C^r} (\boldsymbol{\phi}_\beta \otimes \boldsymbol{e}_\beta^r) \boldsymbol{Q}^T \boldsymbol{\Gamma}_{1_s}^T (\boldsymbol{h} \times \bar{\boldsymbol{t}}) dC^r, \quad (71)$$

$$\bar{\boldsymbol{\pi}}^h = \bar{\boldsymbol{\pi}}^{b,h} + \bar{\boldsymbol{\pi}}^{t,h} = \int_{A^r} (\boldsymbol{\psi} \otimes \boldsymbol{e}_3^r) \boldsymbol{Q}^T \boldsymbol{\Gamma}_{1_s}^T (\boldsymbol{h} \times \bar{\boldsymbol{b}}) dA^r + \int_{C^r} (\boldsymbol{\psi} \otimes \boldsymbol{e}_3^r) \boldsymbol{Q}^T \boldsymbol{\Gamma}_{1_s}^T (\boldsymbol{h} \times \bar{\boldsymbol{t}}) dC^r. \quad (72)$$

Note that both internal and external virtual works are composed of a mid-line and a thickness term. Performing the calculations for the line terms, stress resultants that are energetically conjugates of $\boldsymbol{\eta}^r$, $\boldsymbol{\kappa}^r$, $\boldsymbol{r}$, $\boldsymbol{p}$, $\boldsymbol{r}'$ and $\boldsymbol{p}'$, arise, and are completely equivalent to the generalized stress resultants from Pimenta and Campello [9] (and later with Dasambiagio [11]) (forces ($\boldsymbol{n}$), moments ($\boldsymbol{m}$), generalized bi-shear ($\boldsymbol{\rho}$, $\boldsymbol{\pi}$), and bi-moment ($\boldsymbol{\alpha}$, $\boldsymbol{\beta}$) resultants). For the thickness terms, apart from new contributions to the already defined resultants, there are completely new stress resultants that are energetically conjugated to $\boldsymbol{\theta}, \boldsymbol{\eta}^{r\prime}$, $\boldsymbol{\kappa}^{r\prime}$, $\boldsymbol{r}''$, $\boldsymbol{p}''$.

For the external virtual work, there are external resultants that have already appeared in the aforementioned references ([9], [11]), conjugated to $\boldsymbol{u}$, $\boldsymbol{\theta}$, $\boldsymbol{r}$ and $\boldsymbol{p}$, and also new ones that are conjugated to $\boldsymbol{\eta}^r$, $\boldsymbol{\kappa}^r$, $\boldsymbol{r}'$ and $\boldsymbol{p}'$. Note that, if it is assumed that the external loads act only at the mid-lines (i.e., $\boldsymbol{h} = \boldsymbol{o}$), it follows that $\bar{\boldsymbol{q}}_1{}^{\mathrm{h}} = \bar{\boldsymbol{q}}_2{}^{\mathrm{h}} = \boldsymbol{o}$. Using definitions (34)-(35), (43)-(45) and (58)-(60) in (30)-(32), the weak form of the current formulation becomes

$$\delta W = \int_{\Omega^r} \left[\left(\boldsymbol{\sigma}_1^{m^r} + \boldsymbol{\sigma}_1^{h^r}\right) \cdot \delta\boldsymbol{\varepsilon}_1^r + \boldsymbol{\sigma}_2^{h^r} \cdot \delta\boldsymbol{\varepsilon}_2^r - \left(\bar{\boldsymbol{q}}_1{}^m + \bar{\boldsymbol{q}}_1{}^h\right) \cdot \delta\boldsymbol{d}_1 - \bar{\boldsymbol{q}}_2{}^h \cdot \delta\boldsymbol{d}_2\right] d\Omega^r = 0, \forall\, \delta\boldsymbol{d}_1, \delta\boldsymbol{d}_2 \quad (73)$$

2.3 Tangent operator and constitutive equation

Up to the date of submission of this full paper, the complete expression for the tangent operator is not available. This is the most challenging part of the formulation and is the current priority of the authors. It is not needed to enforce the model´s equilibrium, but plays a pivotal role in the numerical solution, which will be conducted through the finite element method.

Also, no constitutive equation was enforced. Since the deformation gradient is analytically expressed by means of its column-vectors, as done previously in Pimenta and Campello [9], Dasambiagio [11] and Kassab [12], the constitutive equations used therein (namely, from Kirchhoff-Saint-Venant's and Simo-Ciarlet's hyperelastic materials) can be readily used here to calculate the stress resultants. For the tangent operator, some partial derivatives are material-specific.

3 Conclusions

The formulation proposed so far is the result of approximately six months of work in a four-year duration long PhD research. It provides insights into how the formulation will be further developed and is the result of testing of different ways to formalize the model. Since the kinematical description does not depend on the chosen deformation modes, and the equilibrium is written exactly in terms of the Virtual Work Theorem, a robust and versatile description was achieved. Also, the framework enables the introduction of advanced constitutive equations in future steps of this work. The main challenge of the current research is expected to be the obtention of the analytical expression for the tangent operator.

Acknowledgements. This work was supported by FAPESP (São Paulo Research Foundation, grant #2022/15644-0) and CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico, grant #313046/2021-2. The opinions, hypotheses, conclusions and recommendations expressed herein are the sole responsibility of the authors and do not necessarily reflect FAPESP’s and CNPq´s visions.

Authorship statement. The authors hereby confirm that they are the sole liable persons responsible for the authorship of this work, and that all material that has been herein included as part of the present paper is either the property (and authorship) of the authors, or has the permission of the owners to be included here.

References

[1] R. Schardt, “Generalized beam theory—an adequate method for coupled stability problems,” *Thin-Walled Structures*, vol. 19, no. 2–4, pp. 161–180, Jan. 1994, doi: 10.1016/0263-8231(94)90027-2.

[2] R. Gonçalves, M. Ritto-Corrêa, and D. Camotim, “A large displacement and finite rotation thin-walled beam formulation including cross-section deformation,” *Comput Methods Appl Mech Eng*, vol. 199, no. 23–24, pp. 1627–1643, Apr. 2010, doi: 10.1016/j.cma.2010.01.006.

[3] W. Li and H. Ma, “A nonlinear cross-section deformable thin-walled beam finite element model with high-order interpolation of warping displacement,” *Thin-Walled Structures*, vol. 152, p. 106748, Jul. 2020, doi: 10.1016/j.tws.2020.106748.

[4] P. M. Pimenta and T. Yojo, “Geometrically Exact Analysis of Spatial Frames,” *Appl Mech Rev*, vol. 46, no. 11S, pp. S118–S128, Nov. 1993, doi: 10.1115/1.3122626.

[5] P. M. Pimenta and E. M. B. Campello, “Geometrically nonlinear analysis of thin-walled space frames,” Proceedings of the II ECCM (European Conference on Computational Mechanics), 2001, p. 20.

[6] E. M. B. Campello, “Análise não-linear de perfis metálicos conformados a frio,” *São Paulo*, p. 106, 2000.

[7] R. Bebiano, R. Gonçalves, and D. Camotim, “A cross-section analysis procedure to rationalise and automate the performance of GBT-based structural analyses,” *Thin-Walled Structures*, vol. 92, pp. 29–47, Jul. 2015, doi: 10.1016/J.TWS.2015.02.017.

[8] R. Bebiano, D. Camotim, and R. Gonçalves, “GBTul 2.0 − A second-generation code for the GBT-based buckling and vibration analysis of thin-walled members,” *Thin-Walled Structures*, vol. 124, pp. 235–257, Mar. 2018, doi: 10.1016/j.tws.2017.12.002.

[9] P. M. Pimenta and E. M. B. Campello, “A fully nonlinear multi-parameter rod model incorporating general cross-sectional in-plane changes and out-of-plane warping,” *Latin American Journal of Solids and Structures*, vol. 1, no. 1, pp. 119–140, 2003.

[10] P. M. Pimenta, E. S. Almeida Neto, and E. M. B. Campello, “A Fully Nonlinear Thin Shell Model of Kirchhoff-Love Type,” in *New Trends in Thin Structures: Formulation, Optimization and Coupled Problems*, P. M. Pimenta and P. Wriggers, Eds., Vienna: Springer, 2010, pp. 29–58. doi: 10.1007/978-3-7091-0231-2_2.

[11] E. R. Dasambiagio, P. M. Pimenta, and E. M. B. Campello, “A finite strain rod model that incorporates general cross section deformation and its implementation by the Finite Element Method,” in *Mechanics of Solids in Brazil 2009*, H. S. da Costa Mattos and M. Alves, Eds., 1st ed.Rio de Janeiro: Brazilian Society of Mechanical Sciences and Engineering, 2009, pp. 145–168.

[12] M. P. Kassab, E. M. B. Campello, and P. M. Pimenta, “Advances on kinematically exact rod models for thin-walled open-section members: Consistent warping function and nonlinear constitutive equation,” *Comput Methods Appl Mech Eng*, vol. 407, p. 115933, Mar. 2023, doi: 10.1016/j.cma.2023.115933.