Analytical Techniques for Calculating the Work of a Plate

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Abstract. The article analyzes the selection of the deflection function for calculating the functional of the work and presents a matrix approach to simplify this functionality. The problem requiring such a careful approach grew out of the problem of the loss of stability of an infinite plate with an elliptical hole (inclusion) during stretching, where, with an ill-considered choice of deflection functions and irrational calculations, the accumulated error of machine counting ruined the results obtained. The authors have carefully approached the study of this problem and show techniques that allow us to consider the work analytically. These functions and the general approach are also suitable for analytical calculation of potential energy, but this is the topic of a separate article.

Keywords: Work of the Plate , Stability Loss under Tension, Elliptic hole

1 Introduction

This article discusses the problem of how to calculate the work in the problem of stretching an infinite plate with an elliptical hole or inclusion. Generally speaking, for a large class of problems, as well as for problems with curved holes in the general case, it is necessary to calculate the potential energy and work, which are quadratic functionals of deflection. To do this, it suffices to learn how to calculate these functionals for some fairly wide class of deflection functions, from which it will be possible to construct a basis in the future.

In the works of authors Guz et al. [1], Bochkarev and Dal' [2] etc deflection functions were considered that contain the factor $|\omega'|^3$, where ω is a conformal mapping of the exterior of a circle onto the exterior of a curved hole. However, classical works do not explain or derive justification for the choice of the degree, and by the nature of the articles it is clear that no preliminary analysis has been carried out, and the integrals are calculated numerically. We can say that the presence of this factor really seems justified, because the modulus $|\omega'|^2$ itself is included in the determinant of the Jacobi matrix, but it is even more important that $|\omega'|^3$ is a factor that enters the radius of curvature cutout borders. As for the problem with a curvilinear cut or inclusion, due to the complex geometry, the calculation of these functionals leads to very cumbersome calculations, when the accumulated error of the computer calculation destroys the results obtained.

In this article, we consider an elliptical hole (or inclusion), for which it is more convenient to use elliptical coordinates. We will show our reasoning, which greatly simplifies the calculations and obtaining the result analytically, practically without using a computer. We will only talk about the work of forces in the middle plane of the plate on additional displacements that appear due to the tensile load applied at infinity.

Here, it is especially important to choose the correct form of the deflection, in which the integrands will not have a singularity in the denominator, and in the case under consideration it is possible to get rid of it.

The purpose of this article is to show by matrix transformations how to calculate the work of a plate with an elliptical cut (inclusion) for functions containing the factor $|\omega'|^3$ and, in particular, to demonstrate that these integrals do not have singularities at the crack tip (in one of the limiting cases of an ellipse).

2 Analytics

2.1 Notation

Let us introduce some concepts and relations, which we will rely on in what follows. In this paper, we will consider only the case of an elliptical cut/inclusion, for which it is most convenient to use elliptical coordinates:

$$x = c \cdot \cosh\eta \cdot \cos\theta, \quad y = c \cdot \sinh\eta \cdot \sin\theta, \tag{1}$$

c is half the focal length.

Jacobi matrix J of transition to elliptic coordinates:

$$J = \begin{pmatrix} \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \theta} \end{pmatrix}, \qquad J^{-1} = \frac{J^T}{\det J}. \tag{2}$$

Let introduce the deflection function $w = u^{\frac{3}{2}} v(\eta,\theta)$ with the factor u, which is the analogue of $|\omega^{'}|^2$ for the conformal mapping

$$u = \frac{2}{c^2}\det J = \cosh 2\eta - \cos 2\theta, \tag{3}$$

the Hessian matrix H of the deflection function

$$H(w(x,y)) = \begin{pmatrix} \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x \partial y} \\ \frac{\partial^2 w}{\partial x \partial y} & \frac{\partial^2 w}{\partial y^2} \end{pmatrix}, \tag{4}$$

and auxiliary matrices

$$Q_1 = \begin{pmatrix} \sinh 2\eta & \sin 2\theta \\ \sin 2\theta & -\sinh 2\eta \end{pmatrix}, \qquad Q_3 = \begin{pmatrix} 3\sinh 2\eta & u & 0 \\ 3\sin 2\theta & 0 & u \end{pmatrix}. \tag{5}$$

2.2 Work

Consider the symmetric bilinear functional:

$$W(w_1, w_2) = \frac{h}{2}\iint \left[\sigma_{xx}\frac{\partial w_1}{\partial x}\frac{\partial w_2}{\partial x} + \sigma_{yy}\frac{\partial w_1}{\partial y}\frac{\partial w_2}{\partial y} + \tau\left(\frac{\partial w_1}{\partial x}\frac{\partial w_2}{\partial y} + \frac{\partial w_2}{\partial x}\frac{\partial w_1}{\partial y}\right)\right] dxdy, \tag{6}$$

whose quadratic form $W(w,w)$ is the work of forces in the middle plane of the plate.

In this paper, we use stresses in the form proposed in the article of V.M.Malkov and Yu.V.Malkova [3].

The integrand in eq. (6) is a bilinear symmetric mapping with respect to partial derivatives of deflection. Let us decompose the stress tensor into spherical and deviatoric parts:

$$\sigma = \sigma_{sp} + \sigma_{dev} = \begin{pmatrix} \frac{\sigma_{xx}+\sigma_{yy}}{2} & 0 \\ 0 & \frac{\sigma_{xx}+\sigma_{yy}}{2} \end{pmatrix} + \begin{pmatrix} \frac{\sigma_{xx}-\sigma_{yy}}{2} & \tau \\ \tau & -\frac{\sigma_{xx}-\sigma_{yy}}{2} \end{pmatrix},$$

then in matrix form the integrand has the following form:

$$\sigma_{xx}\frac{\partial w_1}{\partial x}\frac{\partial w_2}{\partial x}+\sigma_{yy}\frac{\partial w_1}{\partial y}\frac{\partial w_2}{\partial y}+\tau\left(\frac{\partial w_1}{\partial x}\frac{\partial w_2}{\partial y}+\frac{\partial w_2}{\partial x}\frac{\partial w_1}{\partial y}\right)=$$
$$=\begin{pmatrix}\frac{\partial w_1}{\partial x} & \frac{\partial w_1}{\partial y}\end{pmatrix}\begin{pmatrix}\sigma_{xx} & \tau\\ \tau & \sigma_{yy}\end{pmatrix}\begin{pmatrix}\frac{\partial w_2}{\partial x}\\ \frac{\partial w_2}{\partial y}\end{pmatrix}=\begin{pmatrix}\frac{\partial w_1}{\partial \eta} & \frac{\partial w_1}{\partial \theta}\end{pmatrix}J^{-1}\left(\sigma_{sp}+\sigma_{dev}\right)\left(J^{-1}\right)^T\begin{pmatrix}\frac{\partial w_2}{\partial \eta}\\ \frac{\partial w_2}{\partial \theta}\end{pmatrix}. \quad (7)$$

Here we can see that the matrix σ_{sp} commutes with all matrices. In the case of the transition to elliptic coordinates, significant simplifications are obtained, since, as noted above, $J^{-1}=\frac{J^T}{\det J}$, and besides $J^T\cdot\sigma_{dev}=\sigma_{dev}\cdot J$:

$$J^{-1}\left(\sigma_{sp}+\sigma_{dev}\right)\left(J^{-1}\right)^T=\frac{1}{(\det J)^2}\left(\det J\cdot\sigma_{sp}+\sigma_{dev}\cdot J^2\right). \quad (8)$$

2.3 Prior estimate

A priori, we can show that the integrand eq. (6) will not have a singularity at the crack tip at $\eta=0$ and $\theta=0$ or $\theta=\pi$, if the deflection of a plate with an elliptical cut (inclusion) is sought in the form $w=u^{\frac{3}{2}}\cdot v(\eta,\theta)$. This can be verified not only in the course of direct calculation of the total energy functional, but also with the help of preliminary estimates. For example, for work, the integrand expression eq. (6) with subsequent simplifications eq. (7) and taking into account the Jacobian of the transition to elliptic coordinates can be estimated using the Cauchy-Bunyakovsky-Schwarz inequality:

$$(grad\ w\ )\ \frac{1}{(\det J)^2}\left(\det J\cdot\sigma_{sp}+\sigma_{dev}\cdot J^2\right)\ (grad\ w\ )^T\cdot\det J\leq$$
$$\leq\|grad\ w\ \|\cdot\left\|\frac{1}{\det J}\left(\det J\cdot\sigma_{sp}+\sigma_{dev}\cdot J^2\right)\ (grad\ w\ )^T\right\|.$$

Moreover, if $w\sim u^{3/2}$, then it is quite easy to show that $\|grad\ w\ \|\sim u$. Further, the linear operator $\det J\cdot\sigma_{sp}$ is just expansion, and the linear operator $\sigma_{dev}\cdot J^2$ is the composition of rotations, mirroring and expansion. Rotations and reflection do not change the length of the vector, but according to V.M.Malkov and Yu.V.Malkova [3], the expansions $\sigma_{sp}\sim u^{-1/2}$ and $\sigma_{dev}\sim u^{-3/2}$. The final estimate has the form

$$\|grad\ w\ \|\cdot\left\|\frac{1}{\det J}\left(\det J\cdot\sigma_{sp}+\sigma_{dev}\cdot J^2\right)\ (grad\ w\ )^T\right\|\sim u\cdot u^{-\frac{3}{2}}\cdot u=u^{1/2}.$$

It turns out that the integrand vanishes at the "crack tips" and, moreover, cannot have a singularity at these points.

2.4 Direct calculations

Let us directly verify that when substituting into eq. (6) the functions $w_1=u^{\frac{3}{2}}\cdot v_1(\eta,\theta)$ and $w_2=u^{\frac{3}{2}}\cdot v_2(\eta,\theta)$, we will get rid of the denominator and show how this can be calculated. Recalling notation eq. (5) and differentiating the product, we obtain:

$$\begin{pmatrix}\frac{\partial w_2}{\partial\eta}\\ \frac{\partial w_2}{\partial\theta}\end{pmatrix}=u^{\frac{1}{2}}\begin{pmatrix}3sh2\eta\cdot v_2+u\frac{\partial v_2}{\partial\eta}\\ 3\sin 2\theta\cdot v_2+u\frac{\partial v_2}{\partial\theta}\end{pmatrix}=u^{\frac{1}{2}}\cdot Q_3\cdot\begin{pmatrix}v_2\\ \frac{\partial v_2}{\partial\eta}\\ \frac{\partial v_2}{\partial\theta}\end{pmatrix}.$$

Similarly,

$$\begin{pmatrix}\frac{\partial w_1}{\partial \eta} & \frac{\partial w_1}{\partial \theta}\end{pmatrix} = u^{\frac{1}{2}} \cdot \begin{pmatrix} v_1 & \frac{\partial v_1}{\partial \eta} & \frac{\partial v_1}{\partial \theta}\end{pmatrix} \cdot Q_3^T.$$

Taking into account eq. (8) and the resulting equalities, the integrand in formula eq. (6) takes the form

$$\frac{2}{c^2}\begin{pmatrix} v_1 & \frac{\partial v_1}{\partial \eta} & \frac{\partial v_1}{\partial \theta}\end{pmatrix}\left(\sigma_{sp} \cdot Q_3^T \cdot Q_3 + \frac{1}{\det J} Q_3^T \cdot \sigma_{dev} \cdot J^2 \cdot Q_3\right)\begin{pmatrix} v_2 \\ \frac{\partial v_2}{\partial \eta} \\ \frac{\partial v_2}{\partial \theta}\end{pmatrix}. \tag{9}$$

In the stress formulas in paper [3], σ_{sp} in the denominator contains u to the first power. Since the integrand is multiplied by the Jacobian during the replacement, the denominator disappears in the first term. Then in formula eq. (9) it remains to consider only the second term $2Q_3^T \cdot \sigma_{dev} \cdot J^2 \cdot Q_3/(c^2 \det J)$, and the factor $1/\det J$ vanishes due to the same multiplication by Jacobian. The denominator of the deviatoric part σ_{dev} contains u^3 and the numerator contains $\cos 2n\theta, n = 1, 2, 3$. It is possible to expand $\cos 2n\theta$ in powers of u using the Taylor formula:

$$\cos 2n\theta = T_n\left(\cos 2\theta\right) =$$

$$= T_n(ch2\eta) + T_n^{'}(ch2\eta)\left(\cos 2\theta - ch2\eta\right) + \frac{T_n^{''}(ch2\eta)}{2}\left(\cos 2\theta - ch2\eta\right)^2 + \frac{T_n^{'''}(ch2\eta)}{3!}\left(\cos 2\theta - ch2\eta\right)^3 =$$

$$= T_n(ch2\eta) - T_n^{'}(ch2\eta) \cdot u + \frac{T_n^{''}(ch2\eta)}{2} \cdot u^2 - \frac{T_n^{'''}(ch2\eta)}{3!} \cdot u^3.$$

Here $T_n(x)$ are the Chebyshev polynomials. If $2\sigma_{dev} J^2/c^2$ is expanded in powers of u, then all terms except one do not contain u itself in the denominator:

$$\frac{2}{c^2} \cdot \sigma_{dev} \cdot J^2 = \frac{1}{u} k Q_1 + \dots$$

$$k = -\frac{D + A \cdot m}{2m^2}(1 + m^2 - 2m\cosh 2\eta).$$

Here $m = \frac{a-b}{a+b}$ is axle ratio of the ellipse, A and D notation from Malkov's article of [3] characterizing the load at infinity.

Next, consider the expression $Q_3^T \cdot \sigma_{dev} \cdot J^2 \cdot Q_3$, which is a 3x3 matrix. All elements, except for the one at position [1][1] in the upper left corner, will not have u in the denominator:

$$\left[Q_3^T \cdot \sigma_{dev} \cdot J^2 \cdot Q_3\right]_{11} = \frac{9k}{u}\begin{pmatrix}\sinh 2\eta & \sin 2\theta\end{pmatrix} \cdot Q_1 \cdot \begin{pmatrix}\sinh 2\eta \\ \sin 2\theta\end{pmatrix} =$$

$$= \frac{9k}{u}\sinh 2\eta \cdot \left(\sinh^2 2\eta + \sin^2 2\theta\right) = 9k\ \sinh 2\eta \cdot (\cosh 2\eta + \cos 2\theta).$$

We have shown that the integrand does not contain u in the denominator, and the method proposed above using matrix calculations allows us to significantly simplify the integrand, after which the work integral can be easily calculated.

3 Conclusions

The article shows the rationale for which type of deflection should be taken, for which the integrand can be significantly simplified. Main conclusions:

- a derivation is presented proving that the introduction of a factor into the deflection function $|\omega'|^3$ eliminates the integrand from the singularities at the crack tips, and generally eliminates the functions in the denominator;

- after the transformations done, all integrals can be easily calculated analytically;

- this type of deflection allows you to calculate the potential energy analytically, but this is a topic for a separate article.

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