Logarithmic strain tensor in the positional formulation of FEM

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Abstract. Besides structural engineering, rubber-like materials have been used in several fields such as bioengineering and medicine. Usually, these materials undergo, not only large displacements, but large deformations as well, and this requires a special attention to write a formulation both mathematically and physically consistent. To this end, one must choose a strain measure that, preferably, depends on the initial configuration of the body and that results in the null tensor for a given arbitrary rigid body motion. One of the most used strain measures in this context is the Green strain. However, this measure does not show a physical coherence for the one-dimensional case, as it is known. Thus, this work formulates the problem by using another strain measure, namely, $\ln \mathbf{U}$, where $\mathbf{U}$ is the right stretch tensor. The positional formulation of the finite element method together with the Newton-Raphson procedure are applied for solid analysis. The constitutive equation is supposed to be linear. At the end, a comparison between the numerical results obtained by the Green strain and by $\ln \mathbf{U}$ is performed.

Keywords: logarithmic strain, positional formulation, 3D analysis, St. Venant-Kirchhoff model, hyperelasticity.

1 Introduction

According to Ramezani and Ripin [1], rubber-like materials have several applications such as seals, adhesives, tires, springs, shocks isolators and noise and vibration absorbers. Since they have high deformability, a special care must be taken to formulate its governing equations with physic and mathematic coherence. In this context, the chosen strain measure plays an important role. A well-known strain tensor is the Green strain ${}^{G}\mathbf{E}$, which is widely used by means of the St. Venant-Kirchhoff (SVK) constitutive model. Nevertheless, this constitutive model has some drawbacks for large strain regimes. A common strategy to overcome this limitation is to use the SVK model repeatedly to approximate non-linear stress-strain relations, which is basically a multi-linear procedure. However, Sautter et al. [2] have pointed out that this strategy, while might hold for large tensile strain regimes, is not suitable for large compressive strains, where a nonphysical softening behavior occurs.

Within this scenario, many constitutive relations with alternative strain measures have been proposed, as one can see in Darijani et al. [3], Korobeynikov [4], Annin and Bagrov [5], Korobeynikov et al. [6] and Korobeynikov [7]. Particularly, the Hencky strain, which is the natural logarithm of the left stretch tensor $\mathbf{V}$ ($\ln \mathbf{V}$), and the Hooke's law play a central role. The reason is because Hencky strain forms a work-conjugate pair with the Cauchy stress for an isotropic body, i. e., $\ln \mathbf{V}$ is expected to give plausible physical responses since it conjugates with the true stress. Scientific community does not suspend the study on this strain, as one can see in Le mire et al. [8] and Bertóti [9]. Besides, the material constants of the Hooke's law (Lamé parameters or the Young Modulus E and the Poisson ratio ν) are known for the most materials used in engineering, so its use consists in a considerable advantage.

In this work the strain tensor adopted is ${}^{\ln}\mathbf{E} = \ln \mathbf{U}$. This choice is based on the finite element formulation used here, namely, Positional Formulation of the Finite Element Method (PFFEM), which is intrinsically total lagrangian. Thus, the constitutive equation regarding ${}^{\ln}\mathbf{E}$ (henceforth named as LLOG) is ${}^{R}\mathbf{T} = 2\mu\, {}^{\ln}\mathbf{E} + \lambda(\mathrm{tr}\, {}^{\ln}\mathbf{E})\, \mathbf{I}$, where ${}^{R}\mathbf{T}$ is the work-conjugate stress tensor to ${}^{\ln}\mathbf{E}$, $\mathbf{I}$ is the unit second-order tensor and μ and λ are the Lamé parameters. The use of this constitutive relation together with PFFEM was not studied yet and, as mentioned above, has a great potential to give more reasonable responses than the SKV model. Besides, the present work obtains the indicial notation of $\partial\, {}^{\ln}\mathbf{E}/\partial \mathbf{F}$, where $\mathbf{F}$ is the deformation gradient, using a more general approach than that one (see

Pascon [10]) normally utilized to obtain $\partial^{G}\mathbf{E}/\partial\mathbf{F}$. Actually, the approach of Pascon [10], which is standard in later works using PFFEM, is not able to provide the first derivative of the chosen strain tensor with respect to $\mathbf{F}$ when the strain depends upon $\mathbf{F}$ in a more complex way, such as in the case of ${}^{\ln}\mathbf{E}$.

Worths to mention that, according to Volokh [11], the use of Hooke's law with a work-conjugate stress-strain pair guarantee the existence of the strain energy density, which is a basic assumption in the developing of the section 2.2. At the end of this paper, the results given by SKV and LLOG models are compared.

2 Theoretical fundamentals

2.1 Mathematical preliminaries

Let $\mathcal{V}$, $\mathcal{T}^2$ and $\mathcal{T}^4$ be the set of all vectors, second-order tensors and fourth-order tensors, respectively. If $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the canonical basis of the cartesian coordinate system, then $\mathbf{A} \in \mathcal{T}^2$ and $\mathbb{A} \in \mathcal{T}^4$ are written as $\mathbf{A} = A_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j)$ and $\mathbb{A} = \mathbb{A}_{ijk\ell}(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_\ell)$, respectively. The inner product $\mathbf{A}:\mathbf{B}$ has the following properties

$$\mathbf{A}:(\mathbf{BV}) = (\mathbf{B}^{\mathsf{T}}\mathbf{A}):\mathbf{V} \text{ and } \mathbf{A}:(\mathbf{BCV}) = (\mathbf{B}^{\mathsf{T}}\mathbf{A}\mathbf{V}^{\mathsf{T}}):\mathbf{C}\,, \tag{1}$$

where $\mathbf{B}$, $\mathbf{C}$, $\mathbf{V} \in \mathcal{T}^2$. According to Holzapfel [12], the tensor product $\mathbf{A} \otimes \mathbf{B}$ is the fourth-order tensor such that

$$(\mathbf{A} \otimes \mathbf{B}):\mathbf{V} = (\mathbf{B}:\mathbf{V})\mathbf{A}\,. \tag{2}$$

Moreover, the property

$$\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} \otimes \mathbf{d} = (\mathbf{a} \otimes \mathbf{b}) \otimes (\mathbf{c} \otimes \mathbf{d}) \tag{3}$$

will be useful, with $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d} \in \mathcal{V}$. Next, following Holzapfel [12], there are unit tensors $\mathbb{I}$, $\bar{\mathbb{I}} \in \mathcal{T}^4$ such that

$$\mathbf{A} = \mathbb{I}:\mathbf{A} \text{ and } \mathbf{A}^{\mathsf{T}} = \bar{\mathbb{I}}:\mathbf{A}\,, \tag{4}$$

with

$$\mathbb{I} = \mathbb{I}_{ijk\ell}(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_\ell) = \delta_{ik}\delta_{j\ell}(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_\ell) \text{ and } \bar{\mathbb{I}} = \bar{\mathbb{I}}_{ijk\ell}(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_\ell) = \delta_{i\ell}\delta_{jk}(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_\ell), \tag{5}$$

where $\delta_{(\cdot)(\cdot)}$ is the kronecker delta. Lastly, the Gateaux differential is expressed by

$$D\mathbf{F}(\mathbf{X}_0)[\mathbf{A}] = \left.\frac{d\,\mathbf{F}(\mathbf{X}_0 + \alpha\mathbf{A})}{d\alpha}\right|_{\alpha=0} \tag{6}$$

and

$$\text{if } \begin{cases} \mathbf{F}:\mathcal{T}^2 \to \mathcal{T}^2 \\ \quad \mathbf{X} \mapsto \mathbf{F}(\mathbf{X}) \end{cases}, \text{ then } D\mathbf{F}(\mathbf{X})[\mathbf{A}] = \mathbb{D}:\mathbf{A}, \text{ with } \mathbb{D} = \frac{\partial\mathbf{Y}}{\partial\mathbf{X}} \in \mathcal{T}^4 \text{ and } \mathbf{Y} = \mathbf{F}(\mathbf{X}). \tag{7}$$

2.2 Positional Formulation of the Finite Element Method (PFFEM)

PFFEM has been conceived around 20 years ago and it has shown itself reliable as also accurate. Its recently application in 3D analysis include, e. g., Siqueira and Coda [13] and Pascon [14]. Rather adopting linear displacements as unknown variables, PFFEM uses the nodal positions. In this work, eight nodes hexahedral finite element is used. Following Pascon [14], the usual mappings concerning to a 3D computational implementation are shown in Fig. 1, where ξ, η and ζ represent the fictitious space and $\boldsymbol{\chi}$, $\tilde{\boldsymbol{\chi}}$ and $\bar{\boldsymbol{\chi}}$ are defined such that $\boldsymbol{\chi} = \tilde{\boldsymbol{\chi}} \circ \bar{\boldsymbol{\chi}}^{-1}$.

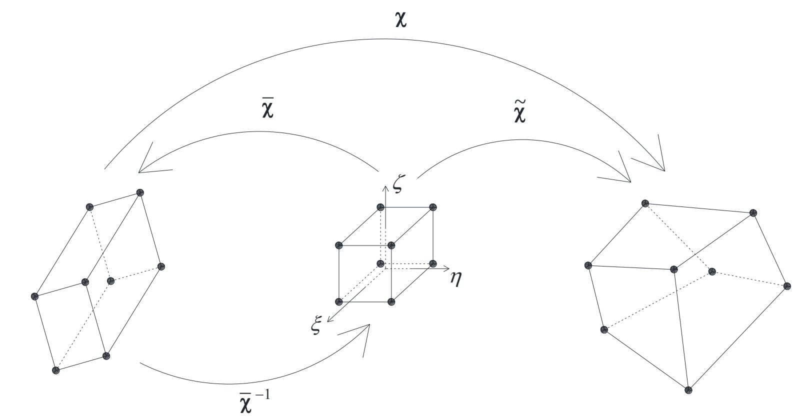


Figure 1. Mappings for 3D computational implementation.

According to Pascon [14],

$$\mathbf{F} = \tilde{\mathbf{F}}\bar{\mathbf{F}}^{-1}, \tag{8}$$

where $\mathbf{F}$, $\tilde{\mathbf{F}}$ and $\bar{\mathbf{F}}$ are the gradients of $\boldsymbol{\chi}$, $\tilde{\boldsymbol{\chi}}$ and $\bar{\boldsymbol{\chi}}$, respectively. Next, the total potential energy Π of a finite element is given by

$$\Pi = \int_{\Omega_0} \psi \, d\mathcal{V}_0 - f_i x_i \,, \tag{9}$$

where Ω_0 indicates the reference configuration, with strain energy density ψ and volume $\mathcal{V}_0$. The term f_i is the force acting on a current nodal direction x_i, with $i = 1, 2, 3, \ldots, 24$. The equilibrium equation for an arbitrary lagrangian work-conjugate stress-strain pair ($\mathbf{T}$, $\mathbf{E}$) is

$$\frac{\partial \Pi}{\partial x_i} = 0 \Rightarrow \int_{\Omega_0} \frac{\partial \psi}{\partial x_i} d\mathcal{V}_0 - f_i = 0 \Rightarrow \int_{\Omega_0} \frac{\partial \psi}{\partial E_{k\ell}} \frac{\partial E_{k\ell}}{\partial x_i} d\mathcal{V}_0 - f_i = 0 \overset{(I)}{\Rightarrow} \int_{\Omega_0} T_{k\ell} \frac{\partial E_{k\ell}}{\partial x_i} d\mathcal{V}_0 - f_i = 0 \Rightarrow$$

$$\Rightarrow \int_{\Omega_0} \left(2\mu E_{k\ell} + \lambda \delta_{k\ell} \operatorname{tr} \mathbf{E}\right) \frac{\partial E_{k\ell}}{\partial x_i} d\mathcal{V}_0 - f_i = 0 \,, \tag{10}$$

where in step (I) one used the definition of hyperelasticity, $T_{k\ell} = \partial\psi/\partial E_{k\ell}$. Now, expression $\partial E_{k\ell}/\partial x_i$ of eq. (10) takes the form

$$\frac{\partial E_{k\ell}}{\partial x_i} = \frac{\partial E_{k\ell}}{\partial F_{no}} \frac{\partial F_{no}}{\partial x_i} \overset{(I)}{=} \frac{\partial E_{k\ell}}{\partial F_{no}} \frac{\partial \left(\tilde{F}_{nw} \bar{F}_{wo}^{-1}\right)}{\partial x_i} = \frac{\partial E_{k\ell}}{\partial F_{no}} \frac{\partial \tilde{F}_{nw}}{\partial x_i} \bar{F}_{wo}^{-1}, \tag{11}$$

where in step (I) one used eq. (8). From eq. (10), let

$$g_i(\mathbf{x}) = \int_{\Omega_0} \left(2\mu E_{k\ell} + \lambda \delta_{k\ell} \operatorname{tr} \mathbf{E}\right) \frac{\partial E_{k\ell}}{\partial x_i} d\mathcal{V}_0 - f_i \tag{12}$$

be a component of a residual vector $\mathbf{g}(\mathbf{x})$. Then, one can uses the Newton-Raphson's Method to solve eq. (10), or $\mathbf{g}(\mathbf{x}) = \mathbf{0}$. Thus, linearizing $\mathbf{g}(\mathbf{x})$ around a known vector $\mathbf{x}_0$, one gives

$$\mathbf{g}(\mathbf{x}_0) + \nabla\mathbf{g}(\mathbf{x}_0)\Delta\mathbf{x} = \mathbf{0} \Rightarrow \Delta\mathbf{x} = -\nabla\mathbf{g}(\mathbf{x}_0)^{-1}\mathbf{g}(\mathbf{x}_0), \tag{13}$$

where the indicial notation of $\nabla\mathbf{g}(\mathbf{x}_0)$ is obtained by eq. (12),

$$\frac{\partial g_i}{\partial x_j}=\int_{\Omega_0}\left\{\left[2\mu\frac{\partial E_{k\ell}}{\partial x_j}+\lambda\,\delta_{k\ell}\left(\frac{\partial E_{11}}{\partial x_j}+\frac{\partial E_{22}}{\partial x_j}+\frac{\partial E_{33}}{\partial x_j}\right)\right]\frac{\partial E_{k\ell}}{\partial x_i}+T_{k\ell}\frac{\partial^2 E_{k\ell}}{\partial x_i\partial x_j}\right\}dV_0\,. \tag{14}$$

For a predefined tolerance *tol*, the convergence of the Newton-Raphson Method is verified by $\|\Delta\mathbf{x}\|/\|\mathbf{x}\|\le tol$. Now, expression $\partial^2E_{k\ell}/\partial x_i\partial x_j$ of eq. (14) takes the form

$$\frac{\partial^2 E_{k\ell}}{\partial x_i\partial x_j}=\frac{\partial}{\partial x_j}\left(\frac{\partial E_{k\ell}}{\partial F_{no}}\frac{\partial \tilde{F}_{nw}}{\partial x_i}\bar{F}^{-1}_{wo}\right)=\frac{\partial \tilde{F}_{nw}}{\partial x_i}\bar{F}^{-1}_{wo}\frac{\partial}{\partial x_j}\left(\frac{\partial E_{k\ell}}{\partial F_{no}}\right)=\frac{\partial \tilde{F}_{nw}}{\partial x_i}\bar{F}^{-1}_{wo}\frac{\partial}{\partial F_{rs}}\left(\frac{\partial E_{k\ell}}{\partial F_{no}}\right)\frac{\partial F_{rs}}{\partial x_j}\overset{(I)}{=}$$

$$\overset{(I)}{=}\frac{\partial \tilde{F}_{nw}}{\partial x_i}\bar{F}^{-1}_{wo}\frac{\partial^2 E_{k\ell}}{\partial F_{no}\partial F_{rs}}\frac{\partial\left(\tilde{F}_{rz}\bar{F}^{-1}_{zs}\right)}{\partial x_j}=\frac{\partial \tilde{F}_{nw}}{\partial x_i}\bar{F}^{-1}_{wo}\frac{\partial \tilde{F}_{rz}}{\partial x_j}\bar{F}^{-1}_{zs}\frac{\partial^2 E_{k\ell}}{\partial F_{no}\partial F_{rs}}\,. \tag{15}$$

where in (I) it was used eq. (8). From eqs. (11) and (15), one concludes that the PFFEM is completely determinated if the terms $\partial E_{k\ell}/\partial F_{no}$ and $\partial^2E_{k\ell}/\partial F_{no}\partial F_{rs}$ of the chosen strain tensor $\mathbf{E}$ are known.

3 Methodology

The methodology of this work to obtain the first derivate of ${}^{\ln}\mathbf{E}$ with respect to the deformation gradient is: 1) write the strain in tensorial notation; 2) apply the Gateaux differential and 3) write the final expression in the format indicated in eq. (7). To exemplify this strategy, one first applies it to the Green strain. Initially, the tensorial notation of ${}^{G}\mathbf{E}$ is

$${}^{G}\mathbf{E}=\mathbf{G}(\mathbf{F})=\frac{1}{2}\left(\mathbf{F}^{\mathsf{T}}\mathbf{F}-\mathbf{I}\right). \tag{16}$$

Next, one uses eq. (6) considering eq. (16),

$$D\mathbf{G}(\mathbf{F})[\mathbf{A}]=\left.\frac{d\,\mathbf{G}(\mathbf{F}+\alpha\mathbf{A})}{d\alpha}\right|_{\alpha=0}=\frac{d}{d\alpha}\left\{\frac{1}{2}\left[(\mathbf{F}+\alpha\mathbf{A})^{\mathsf{T}}(\mathbf{F}+\alpha\mathbf{A})-\mathbf{I}\right]\right\}\Bigg|_{\alpha=0}=\frac{1}{2}\left(\mathbf{F}^{\mathsf{T}}\mathbf{A}+\mathbf{A}^{\mathsf{T}}\mathbf{F}\right). \tag{17}$$

Now, let $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ and $\{\mathbf{d}_1,\mathbf{d}_2,\mathbf{d}_3\}$ be the referential and current bases. Developing $\mathbf{F}^{\mathsf{T}}\mathbf{A}$ of eq. (17) one has

$$\mathbf{F}^{\mathsf{T}}\mathbf{A}\overset{(I)}{=}\mathbb{I}:\left(\mathbf{F}^{\mathsf{T}}\mathbf{A}\right)\overset{(II)}{=}\delta_{kq}\delta_{\ell o}\left(\mathbf{e}_k\otimes\mathbf{e}_\ell\otimes\mathbf{e}_q\otimes\mathbf{e}_o\right):\left[F_{nq}\left(\mathbf{e}_q\otimes\mathbf{d}_n\right)A_{no}\left(\mathbf{d}_n\otimes\mathbf{e}_o\right)\right]\overset{(III)}{=}$$

$$\overset{(III)}{=}\delta_{kq}\delta_{\ell o}F_{nq}A_{no}\left[\left(\mathbf{e}_k\otimes\mathbf{e}_\ell\right)\otimes\left(\mathbf{e}_q\otimes\mathbf{e}_o\right)\right]:\left[\left(\mathbf{e}_q\otimes\mathbf{d}_n\right)\left(\mathbf{d}_n\otimes\mathbf{e}_o\right)\right]\overset{(IV)}{=}\delta_{kq}\delta_{\ell o}F_{nq}A_{no}\left\{\left(\mathbf{e}_q\otimes\mathbf{e}_o\right):\left[\left(\mathbf{e}_q\otimes\mathbf{d}_n\right)\left(\mathbf{d}_n\otimes\mathbf{e}_o\right)\right]\right\}\left(\mathbf{e}_k\otimes\mathbf{e}_\ell\right)\overset{(V)}{=}$$

$$\overset{(V)}{=}\delta_{kq}\delta_{\ell o}F_{nq}A_{no}\left\{\left[\left(\mathbf{d}_n\otimes\mathbf{e}_q\right)\left(\mathbf{e}_q\otimes\mathbf{e}_o\right)\right]:\left(\mathbf{d}_n\otimes\mathbf{e}_o\right)\right\}\left(\mathbf{e}_k\otimes\mathbf{e}_\ell\right)\overset{(VI)}{=}\delta_{kq}\delta_{\ell o}F_{nq}A_{no}\left[\left(\mathbf{e}_k\otimes\mathbf{e}_\ell\right)\otimes\left(\mathbf{d}_n\otimes\mathbf{e}_o\right)\right]:\left[\left(\mathbf{d}_n\otimes\mathbf{e}_q\right)\left(\mathbf{e}_q\otimes\mathbf{e}_o\right)\right]\overset{(VII)}{=}$$

$$\overset{(VII)}{=}\delta_{\ell o}F_{nk}\left(\mathbf{e}_k\otimes\mathbf{e}_\ell\otimes\mathbf{d}_n\otimes\mathbf{e}_o\right):\left[A_{no}\left(\mathbf{d}_n\otimes\mathbf{e}_o\right)\right]\Rightarrow\mathbf{F}^{\mathsf{T}}\mathbf{A}=\delta_{\ell o}F_{nk}\left(\mathbf{e}_k\otimes\mathbf{e}_\ell\otimes\mathbf{d}_n\otimes\mathbf{e}_o\right):\mathbf{A}\,, \tag{18}$$

where in (I) one used eq. (4)a, in (II) eq. (5)a, in (III) eq. (3), in (IV) eq. (2), in (V) eq. (1)a, in (VI) eq. (2) and in (VII) eq. (3). Next, one develops $\mathbf{A}^{\mathsf{T}}\mathbf{F}$ (most of the steps are suppressed since this procedure is similar to the previous one),

$$\mathbf{A}^{\mathsf{T}}\mathbf{F}=\left(\mathbf{F}^{\mathsf{T}}\mathbf{A}\right)^{\mathsf{T}}\overset{(I)}{=}\bar{\mathbb{I}}:\left(\mathbf{F}^{\mathsf{T}}\mathbf{A}\right)\overset{(II)}{=}\delta_{ko}\delta_{\ell q}\left(\mathbf{e}_k\otimes\mathbf{e}_\ell\otimes\mathbf{e}_q\otimes\mathbf{e}_o\right):\left[F_{nq}\left(\mathbf{e}_q\otimes\mathbf{d}_n\right)A_{no}\left(\mathbf{d}_n\otimes\mathbf{e}_o\right)\right]\Rightarrow$$

$$\Rightarrow\mathbf{A}^{\mathsf{T}}\mathbf{F}=\delta_{ko}F_{n\ell}\left(\mathbf{e}_k\otimes\mathbf{e}_\ell\otimes\mathbf{d}_n\otimes\mathbf{e}_o\right):\mathbf{A}\,, \tag{19}$$

where in (I) one used eq. (4)b and in (II) eq. (5)b. Substituting eqs. (18) and (19) in eq. (17) one has

$$D\mathbf{G}(\mathbf{F})[\mathbf{A}]=\frac{1}{2}\left(\delta_{\ell o}F_{nk}+\delta_{ko}F_{n\ell}\right)\left(\mathbf{e}_k\otimes\mathbf{e}_\ell\otimes\mathbf{d}_n\otimes\mathbf{e}_o\right):\mathbf{A}\,. \tag{20}$$

In view of eq. (7), one concludes from eq. (20) that

$$\frac{\partial\,^{\mathrm{G}}\mathbf{E}}{\partial\mathbf{F}} = \frac{1}{2}\left(\delta_{\ell o}F_{nk} + \delta_{ko}F_{n\ell}\right)\left(\mathbf{e}_k \otimes \mathbf{e}_\ell \otimes \mathbf{d}_n \otimes \mathbf{e}_o\right), \tag{21}$$

and $\partial^{\mathrm{G}}E_{k\ell}/\partial F_{no}$ follows directly. This same result can be found in Pascon [10], where a less general approach is used. Now, since $\mathbf{U} = \sqrt{\mathbf{F}^{\mathsf{T}}\mathbf{F}}$, for $^{\ln}\mathbf{E}$ one defines

$$^{\ln}\mathbf{E} = \mathbf{K}(\mathbf{F}) = \ln\left(\sqrt{\mathbf{F}^{\mathsf{T}}\mathbf{F}}\right). \tag{22}$$

Applying eq. (6) and considering eq. (22) one has

$$D\mathbf{K}(\mathbf{F})[\mathbf{A}] = \left.\frac{d\,\mathbf{K}(\mathbf{F}+\alpha\mathbf{A})}{d\alpha}\right|_{\alpha=0} = \frac{d}{d\alpha}\left[\ln\sqrt{(\mathbf{F}+\alpha\mathbf{A})^{\mathsf{T}}(\mathbf{F}+\alpha\mathbf{A})}\right]\Bigg|_{\alpha=0} =$$

$$= \frac{1}{2}\left(\mathbf{F}^{\mathsf{T}}\mathbf{F}\right)^{-1}\left(\mathbf{F}^{\mathsf{T}}\mathbf{A} + \mathbf{A}^{\mathsf{T}}\mathbf{F}\right) \Rightarrow D\mathbf{K}(\mathbf{F})[\mathbf{A}] = \frac{1}{2}\left(\mathbf{F}^{-1}\mathbf{A} + \mathbf{F}^{-1}\mathbf{F}^{-\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{F}\right). \tag{23}$$

Developing $\mathbf{F}^{-1}\mathbf{A}$ of eq. (23) one has (see the procedure to obtain eq. (18) for more details)

$$\mathbf{F}^{-1}\mathbf{A} = \mathbb{I} : \left(\mathbf{F}^{-1}\mathbf{A}\right) = \delta_{kq}\delta_{\ell o}\left(\mathbf{e}_k \otimes \mathbf{e}_\ell \otimes \mathbf{e}_q \otimes \mathbf{e}_o\right) : \left[F^{-1}_{qn}\left(\mathbf{e}_q \otimes \mathbf{d}_n\right)A_{no}\left(\mathbf{d}_n \otimes \mathbf{e}_o\right)\right] \Rightarrow$$

$$\Rightarrow \mathbf{F}^{-1}\mathbf{A} = \delta_{\ell o}F^{-1}_{kn}\left(\mathbf{e}_k \otimes \mathbf{e}_\ell \otimes \mathbf{d}_n \otimes \mathbf{e}_o\right) : \mathbf{A}\,. \tag{24}$$

Developing $\mathbf{F}^{-1}\mathbf{F}^{-\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{F}$ of eq. (23) one gives

$$\mathbf{F}^{-1}\mathbf{F}^{-\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{F} = \left(\mathbf{F}^{\mathsf{T}}\mathbf{A}\mathbf{F}^{-1}\mathbf{F}^{-\mathsf{T}}\right)^{\mathsf{T}} = \bar{\mathbb{I}} : \left(\mathbf{F}^{\mathsf{T}}\mathbf{A}\mathbf{F}^{-1}\mathbf{F}^{-\mathsf{T}}\right) =$$

$$= \delta_{ku}\delta_{\ell q}\left(\mathbf{e}_k \otimes \mathbf{e}_\ell \otimes \mathbf{e}_q \otimes \mathbf{e}_u\right) : \left[F_{nq}\left(\mathbf{e}_q \otimes \mathbf{d}_n\right)A_{no}\left(\mathbf{d}_n \otimes \mathbf{e}_o\right)F^{-1}_{ot}\left(\mathbf{e}_o \otimes \mathbf{d}_t\right)F^{-1}_{ut}\left(\mathbf{d}_t \otimes \mathbf{e}_u\right)\right] \overset{(\mathrm{I})}{=}$$

$$\overset{(\mathrm{I})}{=} \delta_{ku}\delta_{\ell q}F_{nq}A_{no}F^{-1}_{ot}F^{-1}_{ut}\left\{\left(\mathbf{e}_q \otimes \mathbf{e}_u\right) : \left[\left(\mathbf{e}_q \otimes \mathbf{d}_n\right)\left(\mathbf{d}_n \otimes \mathbf{e}_o\right)\left(\mathbf{e}_o \otimes \mathbf{d}_t\right)\left(\mathbf{d}_t \otimes \mathbf{e}_u\right)\right]\right\}\left(\mathbf{e}_k \otimes \mathbf{e}_\ell\right) \overset{(\mathrm{II})}{=}$$

$$\overset{(\mathrm{II})}{=} \delta_{ku}\delta_{\ell q}F_{nq}A_{no}F^{-1}_{ot}F^{-1}_{ut}\left\{\left[\left(\mathbf{d}_n \otimes \mathbf{e}_q\right)\left(\mathbf{e}_q \otimes \mathbf{e}_u\right)\left(\mathbf{e}_u \otimes \mathbf{d}_t\right)\left(\mathbf{d}_t \otimes \mathbf{e}_o\right)\right] : \left(\mathbf{d}_n \otimes \mathbf{e}_o\right)\right\}\left(\mathbf{e}_k \otimes \mathbf{e}_\ell\right) \overset{(\mathrm{III})}{=}$$

$$\overset{(\mathrm{III})}{=} \delta_{ku}\delta_{\ell q}F_{nq}F^{-1}_{ot}F^{-1}_{ut}\left(\mathbf{e}_k \otimes \mathbf{e}_\ell \otimes \mathbf{d}_n \otimes \mathbf{e}_o\right) : \left[A_{no}\left(\mathbf{d}_n \otimes \mathbf{e}_o\right)\right] \Rightarrow$$

$$\Rightarrow \mathbf{F}^{-1}\mathbf{F}^{-\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{F} = F_{n\ell}F^{-1}_{ot}F^{-1}_{kt}\left(\mathbf{e}_k \otimes \mathbf{e}_\ell \otimes \mathbf{d}_n \otimes \mathbf{e}_o\right) : \mathbf{A}\,, \tag{25}$$

where in (I) one used eq. (2), in (II) eq. (1)b and in (III) eq. (2). Substituting eqs. (24) and (25) in eq. (23) one has

$$D\mathbf{K}(\mathbf{F})[\mathbf{A}] = \frac{1}{2}\left(\delta_{\ell o}F^{-1}_{kn} + F_{n\ell}F^{-1}_{ot}F^{-1}_{kt}\right)\left(\mathbf{e}_k \otimes \mathbf{e}_\ell \otimes \mathbf{d}_n \otimes \mathbf{e}_o\right) : \mathbf{A}\,. \tag{26}$$

In view of eq. (7), one concludes from eq. (26) that

$$\frac{\partial\,^{\ln}\mathbf{E}}{\partial\mathbf{F}} = \frac{1}{2}\left(\delta_{\ell o}F^{-1}_{kn} + F_{n\ell}F^{-1}_{ot}F^{-1}_{kt}\right)\left(\mathbf{e}_k \otimes \mathbf{e}_\ell \otimes \mathbf{d}_n \otimes \mathbf{e}_o\right), \quad \text{or} \quad \frac{\partial\,^{\ln}E_{k\ell}}{\partial F_{no}} = \frac{1}{2}\left(\delta_{\ell o}F^{-1}_{kn} + F_{n\ell}F^{-1}_{ot}F^{-1}_{kt}\right). \tag{27}$$

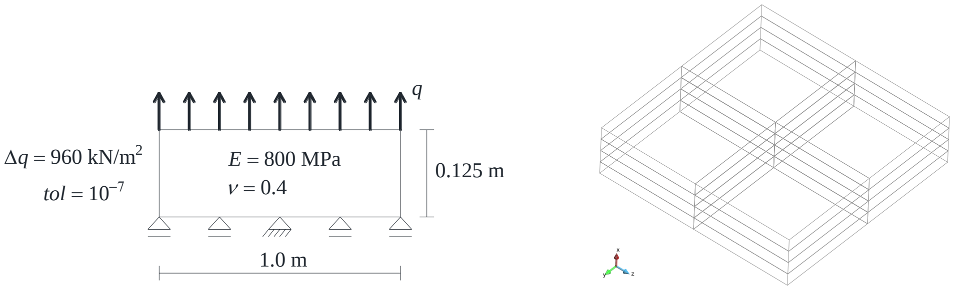
Now, if $\mathbf{Y} = \mathbf{N}(\mathbf{F}) = \mathbf{F}^{-1}$, one can be proved that $\partial Y_{k\ell}/\partial F_{no} = -F^{-1}_{kn}F^{-1}_{o\ell}$. Thus, one derivates eq. (27)b with respect to F_{rs} to obtain

$$\frac{\partial^2\,^{\ln}E_{k\ell}}{\partial F_{no}\partial F_{rs}} = \frac{1}{2}\left[\delta_{\ell o}\left(-F^{-1}_{kr}F^{-1}_{sn}\right) + \mathbb{I}_{n\ell rs}F^{-1}_{ot}F^{-1}_{kt} + F_{n\ell}\left(-F^{-1}_{or}F^{-1}_{st}\right)F^{-1}_{kt} + F_{n\ell}F^{-1}_{ot}\left(-F^{-1}_{kr}F^{-1}_{st}\right)\right] \Rightarrow$$

$$\Rightarrow \frac{\partial^2 \ {}^{\ln}E_{k\ell}}{\partial F_{no} \partial F_{rs}} = \frac{1}{2}\left(-\delta_{\ell o} F_{kr}^{-1} F_{sn}^{-1} + \delta_{nr} \delta_{\ell s} F_{ot}^{-1} F_{kt}^{-1} - F_{n\ell} F_{or}^{-1} F_{st}^{-1} F_{kt}^{-1} - F_{n\ell} F_{ot}^{-1} F_{kr}^{-1} F_{st}^{-1} \right). \tag{28}$$

4 Axial problem

Now, one presents the results of the axial uniformly distributed loading q of a prismatic solid (Fig. 2a) of square base (side 1.0 m) and 0.125 m length. The two plane-symmetry of the Fig. 2a is considered for modelation purposes, leading to the mesh of Fig. 2b. The nonlinear analysis is performed by a fixed incremental load Δq.



(a) (b)

Figure 2. (a) Axial problem data. (b) Mesh used in the analysis (two-plane symmetry is applied).

Fig. 3 shows the results obtained, where λ is the stretching. For the 1D Green strain one has $g(\lambda) = (\lambda^2 - 1)/2$ and for the 1D logarithmic strain $g(\lambda) = \ln \lambda$. The curves for both SKV and LLOG models are concerning to the central node of the top section of the solid in the Fig. 2a. The agreement of the LLOG model is excellent in both tension and compression regimes, as one can see in Fig. 3. However, in the compression regime, the SKV model was unable to give response for $\lambda < 0.6$, as indicated in Fig. 3. In tension regime, SKV model shows great results.

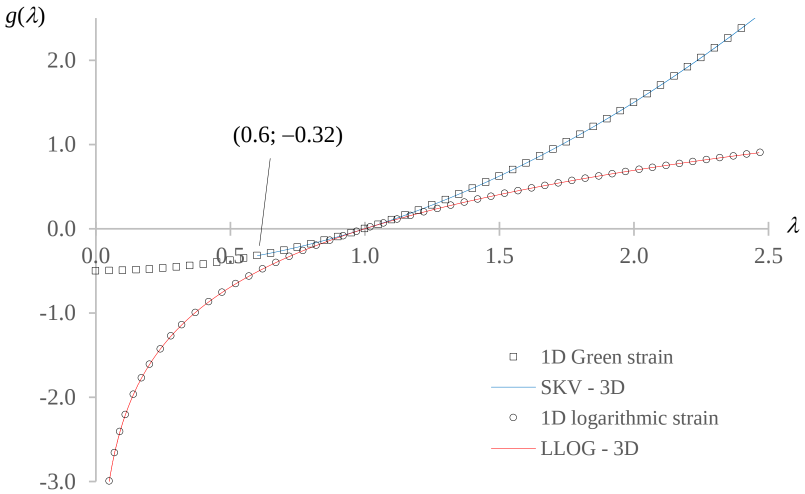


Figure 3. Axial problem results.

5 Conclusions

Using a more general approach to obtain the indicial notation of the derivative of a strain measure with respect to the deformation gradient, this work presents and implements PFFEM considering the logarithmic strain tensor ($\ln \mathbf{U}$) by means of a linear constitutive relation between it and its work-conjugate stress tensor. The results of this constitutive equation were exceptional for the axial problem. In the compression regime, it was able to provide responses for very small stretchings, unlikely SKV model, which has stopped in a stretching around 0.6. Therefore, the LLOG model proved itself worthy of further investigations.

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