

Structural Dynamic Analysis Considering a Locally-defined Time Integration Procedure

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Abstract. This paper addresses the use of a locally-defined time-marching methodology for the dynamic analysis of frame structures. The discussed formulation allows specifying time integration parameters at an element level, enabling intended numerical features, such as numerical damping, to be locally applied, providing a much more versatile approach. Additionally, these locally-defined time integration parameters may also be established following the inherent properties of each element of the discretized model, allowing optimized definitions to be carried out and enhanced accuracy to be provided. The presented technique is unconditionally stable, second-order accurate, and truly self-starting, and it allows adaptive controllable algorithm dissipation. At the end of the manuscript, numerical examples are presented, illustrating the effectiveness of the discussed methodology.

Keywords: time marching, dynamic analysis, local parameters, controllable dissipation.

1 Introduction

Numerical methods serve as fundamental tools for solving intricate problems encountered in various scientific and engineering domains. Specifically, in structural dynamics, numerical methods are employed to ascertain the stresses, strains, and displacements experienced by a system when subjected to arbitrary loads or initial conditions. In this context, the finite element method [1] is frequently utilized for spatial discretization, while the dynamic response can be obtained using two typical standard approaches. The first approach involves the utilization of the mode superposition method, whereby the final response is determined as the summation of the respective vibration modes. The second approach, known as direct integration, avoids any transformation of equations into an alternate form, thereby enabling the direct computation of the response through time-marching or step-by-step methods. The scientific literature encompasses numerous classical implicit [2-5] and explicit [6-9] methods for structural dynamics analysis, and a comprehensive review of these methods can be found in [10-13].

For problems in structural dynamics, the inclusion of numerical damping in time-marching techniques becomes crucial due to the presence of spurious high-frequency modes introduced by spatial discretization. Controllable dissipation methods are necessary to suppress the contribution of these high-frequency modes while preserving the accuracy of low-frequency modes, which are responsible for generating the correct solution response. In this context, several methods have been proposed to introduce numerical dissipation and eliminate spurious high frequencies. The classical Newmark method [2] is widely employed and accepted in both practical engineering and scientific communities. However, when numerical damping is applied, the Newmark method is only first-order accurate.

Based on the framework established by Soares [14, 15], this study discusses a locally defined implicit time integration procedure. In this method, users have the flexibility to selectively apply numerical dissipation to specific elements. To achieve this, an additional property, referred to as the numerical dissipation parameter (denoted as a_e), is assigned to each element of the spatial discretization, akin to the provision of physical properties. This approach enables the inclusion of controllable, locally-defined numerical dissipation in the analyses.

The utilization of adaptively defined time-marching procedures, both on a global scale [14, 16] and on a local scale [15, 17, 18], is not novel within the field of structural dynamics, wave propagation analyses, and other transient

applications [19]. However, this study presents a novel methodology where the user has the ability to locally define the properties of the time solution procedure in a manner analogous to defining the standard properties of the spatially discretized elements within the model, as well as to separately consider the axial and the transverse behavior of the frame element when defining these properties. Consequently, the user can selectively specify the elements in which numerical damping should be considered, along with its desired intensity. Subsequently, the technique is adaptively developed based on these user-defined specifications.

2 Governing equations and time marching procedure

The dynamic model is defined by the following governing system of equations, as presented in [20]:

$$\mathbf{M}\ddot{\mathbf{U}}(t) + \mathbf{C}\dot{\mathbf{U}}(t) + \mathbf{K}\mathbf{U}(t) = \mathbf{F}(t), \quad (1)$$

where $\mathbf{F}(t)$ represents the load vector; \mathbf{M} , \mathbf{C} , and \mathbf{K} represent the mass, damping, and stiffness matrices, respectively; and $\mathbf{U}(t)$, $\dot{\mathbf{U}}(t)$, and $\ddot{\mathbf{U}}(t)$ denote the displacement, velocity, and acceleration vectors, respectively. The initial conditions of the model are given by $\mathbf{U}^0 = \mathbf{U}(0)$ and $\dot{\mathbf{U}}^0 = \dot{\mathbf{U}}(0)$, where \mathbf{U}^0 and $\dot{\mathbf{U}}^0$ represent the initial displacement and initial velocity vectors, respectively.

By considering a constant time-step Δt (i.e., $t^{n+1} = t^n + \Delta t$) and integrating Eq. (1) over time, at the element level (denoted by subscript e), we can express this equation as follows:

$$\mathbf{M}_e \int_{t^n}^{t^{n+1}} \ddot{\mathbf{U}}_e(t) dt + \mathbf{C}_e \int_{t^n}^{t^{n+1}} \dot{\mathbf{U}}_e(t) dt + \mathbf{K}_e \int_{t^n}^{t^{n+1}} \mathbf{U}_e(t) dt = \int_{t^n}^{t^{n+1}} \mathbf{F}_e(t) dt. \quad (2)$$

The integrals on the left-hand side of Eq. (2) can be computed as [14]:

$$\int_{t^n}^{t^{n+1}} \ddot{\mathbf{U}}_e(t) dt = \dot{\mathbf{U}}_e^{n+1} - \dot{\mathbf{U}}_e^n, \quad (3a)$$

$$\int_{t^n}^{t^{n+1}} \dot{\mathbf{U}}_e(t) dt = \mathbf{U}_e^{n+1} - \mathbf{U}_e^n, \quad (3b)$$

$$\int_{t^n}^{t^{n+1}} \mathbf{U}_e(t) dt = \Delta t \mathbf{U}_e^n + \frac{1}{2} \alpha_e \Delta t^2 \dot{\mathbf{U}}_e^n + \frac{1}{2} \gamma_e \Delta t^2 \ddot{\mathbf{U}}_e^{n+1}, \quad (3c)$$

where α_e and γ_e are the parameters of the method, calculated for each element, and the superscripts n and $n+1$ denote the previous and the current time steps, respectively. The displacement \mathbf{U}^{n+1} can be defined as follows:

$$\mathbf{U}^{n+1} = \mathbf{U}^n + \frac{1}{2} \Delta t (\dot{\mathbf{U}}^n + \dot{\mathbf{U}}^{n+1}). \quad (4)$$

Given equations (3) and (4), equation (2) can be redefined as a straightforward recursive relationship:

$$\mathbf{E}_{1e} \dot{\mathbf{U}}_e^{n+1} = \int_{t^n}^{t^{n+1}} \mathbf{F}_e(t) dt + \mathbf{E}_{2e} \dot{\mathbf{U}}_e^n - \Delta t \mathbf{K}_e \mathbf{U}_e^n. \quad (5)$$

where:

$$\mathbf{E}_{1e} = \left(\mathbf{M}_e + \frac{1}{2} \Delta t \mathbf{C}_e + \frac{1}{2} \Delta t^2 (\gamma_e \mathbf{K}_e) \right) \quad (6a)$$

$$\mathbf{E}_{2e} = \left(\mathbf{M}_e - \frac{1}{2} \Delta t \mathbf{C}_e - \frac{1}{2} \Delta t^2 (\alpha_e \mathbf{K}_e) \right) \quad (6b)$$

Therefore, once the assembly process is completed, velocities can be calculated according to Eq. (5), and, subsequently, displacements can be evaluated using Eq. (4). The proposed technique is truly self-starting, relying only on displacement-velocity relationships, eliminating the need for acceleration calculations and initial condition procedures.

In the proposed methodology, for each element, two maximum natural frequencies are calculated (i.e., ω_{et}^{max} and ω_{eb}^{max}), and parameters γ_e and α_e are computed as function of these natural frequencies. Therefore, there will be two sets of γ_e and α_e parameters per element. The values of γ_e and α_e should be multiplied by their respective corresponding terms in the element stiffness matrix, according to Eqs. (6a-b).

The integration parameters α_e and γ_e are locally determined and computed based on the element properties, the time-step Δt , the value of the numerical dissipative property a_e , and the maximum natural frequency ω_e^{max} (which may be defined by ω_{et}^{max} or ω_{eb}^{max}). The methodology allows the user to selectively activate numerical dissipation in specific structural elements, thus, wherever numerical dissipation is desired, $a_e > 0$ is adopted; otherwise, $a_e = 0$ is assigned. When $a_e = 0$, the following definitions for the time integration parameters are considered:

$$\gamma_e = \frac{1}{2} \tanh\left(\frac{1}{4} \omega_e^{max} \Delta t\right), \quad (7a)$$

$$\alpha_e = 1 - \gamma_e. \quad (7b)$$

Otherwise, if $a_e > 0$, γ_e and α_e are defined as follows:

$$\gamma_e = \frac{1}{2} + \frac{3}{2} \tanh(a_e \omega_e^{max} \Delta t), \quad (8a)$$

$$\alpha_e = 2(2\gamma_e)^{\frac{1}{2}} - \gamma_e - 1, \quad (8b)$$

where ω_{et}^{max} and ω_{eb}^{max} represent the maximum natural frequencies for the truss ($\mathbf{K}_{et} \subset \mathbf{K}_e$ and $\mathbf{M}_{et} \subset \mathbf{M}_e$) and beam ($\mathbf{K}_{eb} \subset \mathbf{K}_e$ and $\mathbf{M}_{eb} \subset \mathbf{M}_e$) elements, respectively, calculated as the square root of the largest eigenvalues obtained from the generalized eigenvalue problems

$$\mathbf{K}_{et} \Phi = (\omega_{et}^{max})^2 \mathbf{M}_{et} \Phi, \quad (9a)$$

$$\mathbf{K}_{eb} \Phi = (\omega_{eb}^{max})^2 \mathbf{M}_{eb} \Phi. \quad (9b)$$

The basic steps of the discussed solution procedure are presented in Table 1.

Table 1. Solution algorithm

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1. Initialize \mathbf{U}^0 and $\dot{\mathbf{U}}^0$, and select a time step Δt for the analysis;
 2. For each element of the model:
 - 2.1. Input the numerical damping property a_e ;
 - 2.2. Compute \mathbf{K}_e (defined by $\mathbf{K}_{et} \cup \mathbf{K}_{eb}$), \mathbf{M}_e (defined by $\mathbf{M}_{et} \cup \mathbf{M}_{eb}$), and \mathbf{C}_e , at local coordinates;
 - 2.3. Compute the max. natural frequencies ω_{et}^{max} and ω_{eb}^{max} : Eqs. (9a-b);
 - 2.4. Compute γ_e and α_e for each max. natural frequency: if ($a_e = 0$): Eqs. (7a-b); otherwise: Eqs. (8a-b)
 - 2.5. Establish $\gamma_e \mathbf{K}_e$ as $(\gamma_{et} \mathbf{K}_{et}) \cup (\gamma_{eb} \mathbf{K}_{eb})$ and $\alpha_e \mathbf{K}_e$ as $(\alpha_{et} \mathbf{K}_{et}) \cup (\alpha_{eb} \mathbf{K}_{eb})$;
 - 2.6. Establish the local matrices at global coordinates;
 - 2.7. Compute the auxiliary matrices \mathbf{E}_{1e} and \mathbf{E}_{2e} : Eqs. (6a-b), and assemble them (as well as \mathbf{K}_e);
 3. Compute $\mathbf{E}_1 = \mathbf{LDL}^T$;
 4. For each time step:
 - 4.1. Compute $\mathbf{R} = \int_{t^n}^{t^{n+1}} \mathbf{F}(t) dt$ (as, for instance, $\mathbf{R} = \frac{1}{2} \Delta t (\mathbf{F}^n + \mathbf{F}^{n+1})$);
 - 4.2. Compute the velocity vector: $\mathbf{LDL}^T \dot{\mathbf{U}}^{n+1} = \mathbf{R} + \mathbf{E}_2 \dot{\mathbf{U}}^n - \Delta t \mathbf{K} \mathbf{U}^n$;
 - 4.3. Compute the displacement vector: Eq. (4);
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3 Numerical Application

In this section, a numerical example is discussed to illustrate the performance of the reported method. We used the MATLAB programming language (version R2018a) to perform the computational simulations. A heterogeneous rod, subjected to an axial force, is considered, and the computed results are compared with those provided by the Newmark [2] and Bathe [5] methods. A sketch of the model is presented in Figure 1. As illustrated in this figure, the rod is fixed at its left border (Point A) and subjected to a prescribed axial traction acting on its right border (Point C), which is defined by:

$$P(t) = F_0 \sin\left(\frac{\pi t}{T}\right) [H(t) - H(t - T)], \quad (10)$$

where $F_0 = 1 \times 10^3 [N]$ and $T = 0.01 [s]$ describe the amplitude and the duration of the applied traction, respectively, and $H(\cdot)$ stands for the Heaviside function. The length of the rod is defined by $L = 1.0 [m]$, and two equal-sized subdomains compose the referred heterogeneous model, as shown in Fig.1. In this case, the left subdomain of the rod (subdomain AB) is characterized by a material with a p-wave propagation velocity defined by $c_{AB} = 10 [m/s]$, and different materials are utilized to characterize the right subdomain of the model (subdomain BC). Thus, in this example, the following values are considered for the p-wave propagation velocity of subdomain BC: (i) $c_{BC} = 10 [m/s]$ (model 1, homogeneous rod), (ii) $c_{BC} = 20 [m/s]$ (model 2); (iii) $c_{BC} = 30 [m/s]$ (model 3); (iv) $c_{BC} = 40 [m/s]$ (model 4); 50 linear finite elements of equal length are employed for the spatial discretization and $\Delta t = 5 \times 10^{-2} T [s]$ is adopted. For this model, the analytical answer for the axial displacements is found in [21].

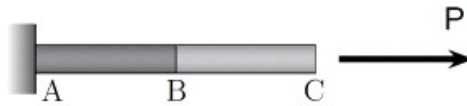


Figure 1. Sketch of the heterogeneous rod.

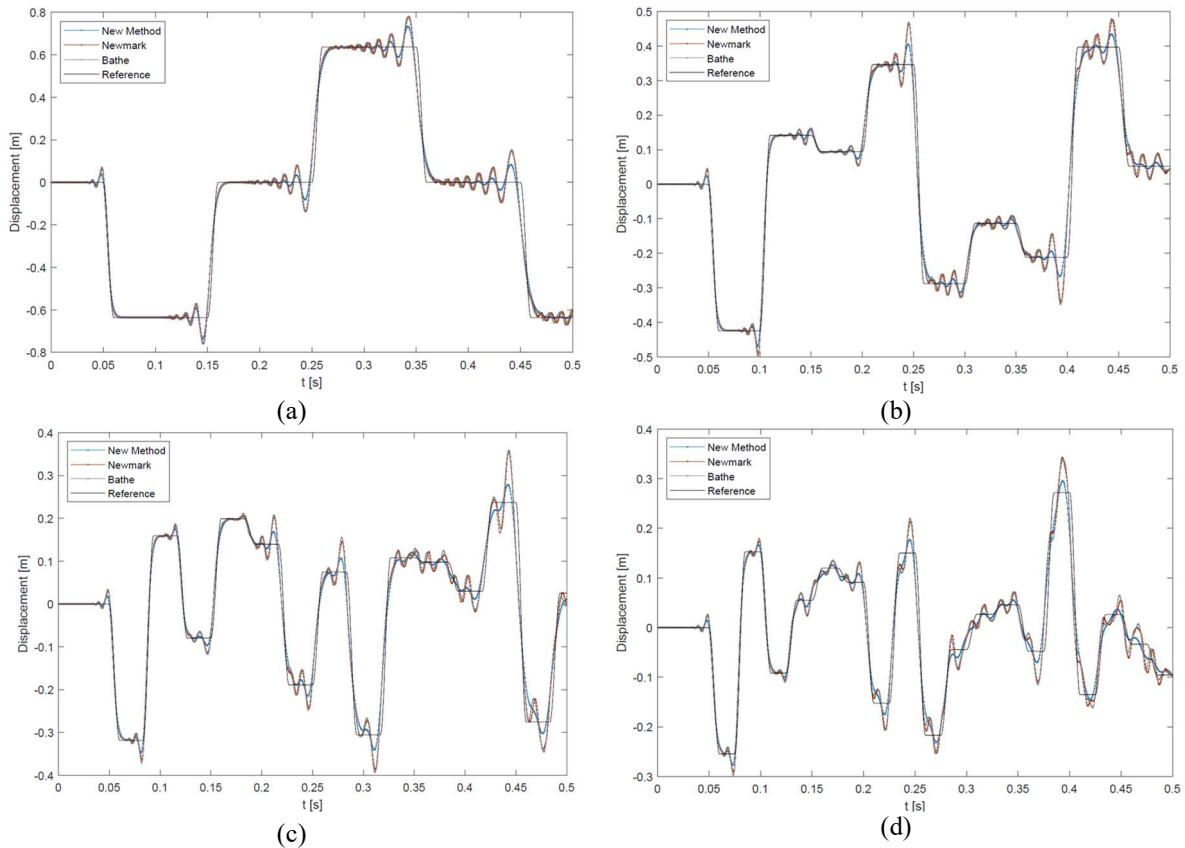


Figure 2. Axial displacements at the middle of the rod for: (a) $c_{BC}/c_{AB} = 1$; (b) $c_{BC}/c_{AB} = 2$; (c) $c_{BC}/c_{AB} = 3$; (d) $c_{BC}/c_{AB} = 4$.

Taking into account the discussed approach, $a_e = 0,1$ was adopted only for the segment AB of the model (located near the support), while $a_e = 0$ was considered for all other elements (segment BC). To calculate the relative errors of the computed results and compare the performance of the different time integration procedures, the following expression is utilized:

$$Erro = \left[\frac{\sum_{n=1}^N (u^n - u_a(t^n))^2}{\sum_{n=1}^N (u_a(t^n))^2} \right]^{\frac{1}{2}} \quad (11)$$

in which u stands for the computed time-history response of a selected degree of freedom, u_a corresponds to its analytical counterpart, and N represents the total number of time steps in the analysis.

In Figure 2, the time history results for the axial displacements at the center of the rod are depicted, considering the referred time integration procedures and $\Delta t = 5 \times 10^{-4}$ s. In Table 2, the computed relative errors for these results are presented. As one may observe, the proposed novel formulation provides exceptionally more accurate responses than the referred standard techniques, even yielding much better results than the selected composite time integration technique, which considers two solver procedures per time step.

Table 2. Relative errors for the displacements at the middle of the rod

Method	c_{BC}/c_{AB}			
	1	2	3	4
New	$0.16 \cdot 10^{-2}$	$0.37 \cdot 10^{-2}$	$0.55 \cdot 10^{-2}$	$0.67 \cdot 10^{-2}$
Newmark	$0.58 \cdot 10^{-2}$	$0.72 \cdot 10^{-2}$	$0.96 \cdot 10^{-2}$	$1.11 \cdot 10^{-2}$
Bathe	$0.36 \cdot 10^{-2}$	$0.62 \cdot 10^{-2}$	$0.90 \cdot 10^{-2}$	$1.01 \cdot 10^{-2}$

4 Conclusions

This work discusses an alternative time marching procedure for structural dynamics analysis. The discussed method, which is based on displacement and velocity relations, eliminates the need for computing accelerations (which may be post-processed, if requested), making it simple and efficient in terms of computational effort. The method is truly self-starting, and it allows for controllable numerical dissipation at the element level, giving the user the flexibility to introduce numerical dissipation in selected elements and adjust its intensity.

The numerical dissipation property a_e can be interpreted as an additional physical property of the element, similar to the moment of inertia, mass density, or Young's modulus. The method ensures stability and maintains second-order accuracy when numerical dissipation is introduced, a characteristic that is not always present in classical methods like the Newmark method. In Section 3, results are presented to demonstrate the effectiveness of the proposed method. As illustrated, the new technique may outperform the computationally demanding Bathe method [5], offering improved performance.

Acknowledgments. The authors would like to acknowledge the support and collaboration of COPPE - UFRJ (Federal University of Rio de Janeiro) and UFJF (Federal University of Juiz de Fora) in the development of this scientific paper. Additionally, the authors would like to express their gratitude for the financial support provided by CNPq (National Council for Scientific and Technological Development).

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