Nonlinear Dynamics in Graded Waterbomb Origami Tubes

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Abstract. The graded Waterbomb origami tube, an extension of the classic Waterbomb pattern, offers a versatile platform with potential applications in diverse fields. This periodic 3D structure can exhibit wave-like behavior along its longitudinal axis, making it intriguing for applications ranging from metamaterials to medical implants. Recent work has connected its wave properties to a discrete 2D dynamical system. We extend this concept by investigating the impact of introducing gradation into unit cells along tessellation lines. This introduces a non-autonomous nonlinear iterated map that generates ordered and disordered origami tubes based on deterministic and stochastic rules. In this study, we explore these dynamics and report their distinctive properties.

Keywords: Origami geometry, Dynamical systems, Metamaterials

1 Introduction

Origami, the traditional Japanese art of paper folding, has transcended its recreational origins to become a platform of intricate mathematical modeling and simulations [1, 2]. This evolution has enabled the accurate exploration of folding dynamics [3–6], revealing complex patterns [7, 8]. Beyond aesthetics, origami's geometric and mechanical properties, alongside recent material science advancements, have positioned it as a tool for innovative smart system design [9–13].

The *Waterbomb* origami, a foundational structure, is remarkably versatile, forming the basis of intricate geometric shapes [14–18]. This adaptability extends to crafting 3D tubes (Fig. 1 left). Introducing geometric gradation amplifies its adaptability, potentially broadening its functionalities. Despite its potential, graded Waterbomb origami remains underexplored. This work investigates graded Waterbomb's link to a 5-dimensional dynamical system, unveiling fresh insights into its intricate behavior.

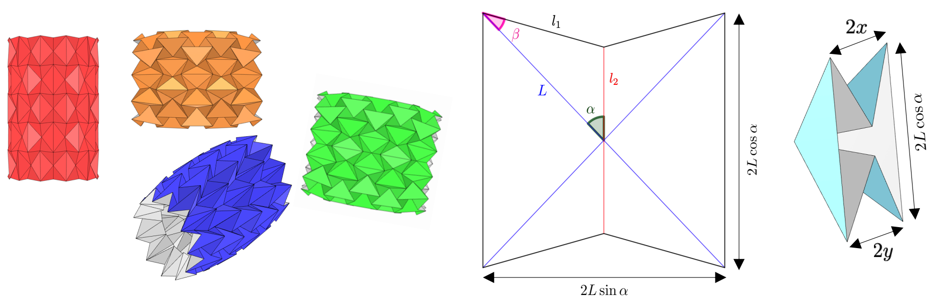

Figure 1. 3D folded configurations of a tubular structure derived from the Waterbomb origami (left), showcasing the unfolded unit cell (center) alongside the folded 3D unit cell (right).

2 Mathematical modeling

Recent investigations by Imada and Tachi [19, 20] have illuminated a link between the wave properties of the Waterbomb tube and a discrete 2D dynamical system. In a further exploration of this connection, our study delves into the influence of gradation within unit cells along tessellation lines. This introduction of a controlled variation in the properties of the unit cells leads to the formulation of a non-autonomous nonlinear iterated map:

$$
\begin{aligned}
x_{m+1} &= f(x_m, y_m, \alpha_m, \beta_m, L_m) \\
y_{m+1} &= g(x_m, y_m, \alpha_m, \beta_m, L_m) \\
\alpha_{m+1} &= p(\alpha_m, \beta_m, L_m) \\
\beta_{m+1} &= q(\alpha_m, \beta_m, L_m) \\
L_{m+1} &= s(L_m, \delta, T)\,.
\end{aligned}
\tag{1}
$$

Here, x_m and y_m denote state variables characterizing the m-th unit cell configuration (Fig 1 right), while α_m, β_m, and L_m are parameters associated with the m-th unfolded unit cell (Fig 1 center). Functions f and g govern the evolution of x_m and y_m respectively, p and q dictate the progression of α_m and β_m, and s orchestrates the scaling of L_m, with δ representing the gradation level parameter governing the change in L_m, while T signifies the gradation period. The gradation function ushers controlled variation in unit cell properties, and diverse patterns (Fig. 2) emerge from it, uniquely impacting the origami tube's behavior.

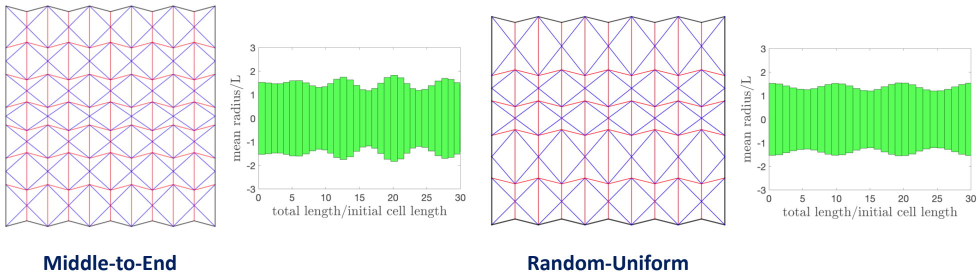

Figure 2. Waterbomb origami crease patterns showcasing two distinct gradation structures, alongside corresponding cross-sections depicting possible configurations for the 3D folded tubes.

Two specific gradation patterns are investigated: Middle-to-End (deterministic) and Uniform Random (stochastic). The former introduces an ordered tube, where the gradation pattern adheres to a predetermined sequence. The latter results in a disordered tube, introducing randomness into the gradation process and thereby distinctively influencing the tube's characteristics. Through a comprehensive analysis of these two types of tubes, we glean valuable insights into the graded Waterbomb origami's underlying dynamics and its potential practical applications.

For an in-depth understanding of the mathematical foundations and deductions underlying the nonlinear iterated map, as well as the detailed derivation of the map equations governing the dynamics of the graded Waterbomb origami tubes, we refer readers to our companion paper [21]. This work provides a comprehensive exploration of the intricate relationship between state variables, parameters, and functions that drive the origami's evolution, unraveling the emergence of ordered and disordered origami tubes. It underscores the map equations as a crucial bridge connecting unit cell gradation and the resulting origami structure dynamics.

3 Results and discussion

This section showcases the results obtained from our investigation into the Waterbomb discrete dynamics within the realms of the End-to-Middle and Uniform-Random cases. Our focus shifts towards the phase portraits displayed in Figure 3, a visual medium that encapsulates the intricacies inherent in these distinct scenarios. Our exploration leads to the identification of two principal types of solutions. Firstly, the finite-size solutions are characterized by trajectories that progressively expand, amassing energy $E_m \sim x_m^2 + y_m^2$ until a critical threshold triggers a swift and often explosive transition to geometrically incompatible configurations, indicative of a sudden loss of stability. In contrast, the wave-like solutions manifest as oscillatory behaviors between the upper and lower vertices of the unit cell. This rhythmic exchange of energy upholds a dynamic equilibrium, effectively restraining the system from venturing into geometrically unsuitable configurations.

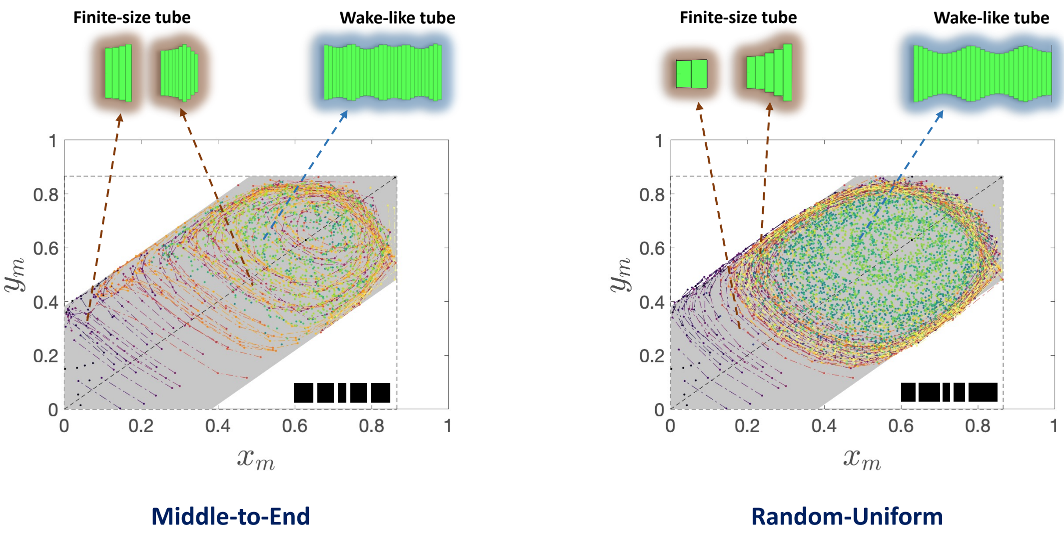

Figure 3. Phase portraits for $(\alpha, \beta, N, \delta, T) = (\pi/3, \pi/4, 8, 1/100, 7)$.

In the context of the End-to-Middle pattern, finite-size orbits stand out as the dominant dynamic behavior. However, in the Uniform-Random pattern, an almost even distribution of both wave-like and finite-size orbits is observed. Notably, in the Uniform-Random case, the wave-like orbits exhibit a distinct irregularity in comparison to their counterparts in the classic Waterbomb configuration. This subtle yet noteworthy disparity in wave-like behavior indicates a significant diversion from conventional dynamics, highlighting the intricate influence of gradation on the origami tube's behavioral nuances.

The insights gleaned from these phase portraits shine a spotlight on the intricate behaviors that surface when gradation regulations are introduced, offering deep comprehension into the dynamic motifs exhibited by origami tubes. This analytical exploration lays bare the captivating dynamics governing the ordered and disordered origami tubes within these predefined contexts, ultimately advancing our comprehension of the underlying mechanisms.

4 Conclusions

In essence, our investigation into the dynamics of graded Waterbomb origami tubes underscores the influence of incorporating gradation within their unit cells. This pioneering approach introduces a novel nonlinear iterated map, which in turn, unveils the intricate underpinnings behind the emergence of both ordered and disordered origami tubes. This study contributes to our comprehension of origami structures enriched with gradation, while concurrently broadening the potential horizons for practical applications spanning diverse domains.

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