Regularization of complex flexibility of layered models of railway track

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Abstract. In this contribution, a regularization technique of the function representing the complex flexibility of layered models of ballasted railway track is proposed in a way that allows one to keep the general steps from the one-layer model and analyze in depth other models that are inherently more realistic. With this regularization, it is also possible to more precisely analyze how damping can negatively affect the instability of two or more moving proximate masses. The results are presented in a dimensionless form, and it is demonstrated that the proposed technique works reasonably well.

Keywords: complex flexibility, moving mass, layered models, contour integration, integral transform.

1 Introduction

Layered models of ballasted railway track are useful computational models that are still very popular due to their computational efficiency, ease of implementation and results analysis (Dimitrovová [1]); and good approximation of reality, that is, they provide reasonable agreement with results obtained on detailed models, as shown in Rodrigues and Dimitrovová [2]. Using semianalytical methods, most of the results can be obtained in an accurate way, Dimitrovová [3,4]. These methods also enable instability analysis of moving inertial objects, e.g., moving masses. The strong interaction between moving masses has been reported in Dimitrovová [5,6]. Ease of implementation in most of these works was facilitated by so-called induced frequencies. However, only for a one-layer model, which in fact corresponds to the Winkler-Pasternak beam, can such frequencies be determined accurately for the undamped case and subcritical velocities, which is the situation where the interaction is most apparent. In the two- and three-layer models, there are no real induced frequencies that can be used for such an analysis. Obviously, time series can be obtained by numerical inverse Laplace transform, but this is not the way to draw general conclusions.

The reason for the non-existence of real induced frequencies is the fact that the complex flexibility of the two- and three-layer model is never real-valued. In the two-layer model the imaginary part of the complex flexibility in undamped case and subcritical velocity range is practically negligible and thus can be ignored. In the three-layer model this is no longer possible.

In more detail, in Dimitrovová [4] it was demonstrated that the complex flexibility of the layered models exhibits discontinuities. These discontinuities are bounded, so the contour integration can be performed when appropriate branch cuts are introduced, but the discontinuities cause that the so-called frequency lines can be interrupted and induced frequencies in some cases do not exist. A discontinuity is always related to a real root of the characteristic equation of the flexibility function. Luckily, such discontinuities always occur only within the region with stable induced frequencies, therefore the instability judgment is not compromised. However, for analysis of the dynamic interaction between proximate masses, the lack of induced frequencies can affect the existence of a reliable closed form solution. By analyzing the time series, it is noticed that when the velocity is changed smoothly, then the deflection shapes in subcritical velocity range are also changing smoothly. This happens independently on the fact whether the frequency line is interrupted or not. Meaning, when there are induced frequencies, then the time series can be approximately taken as sum of residues, when there are no induced frequencies, then the time series are obtained by integration along the branch cuts, but visually resemble the case determined by the sum of residues, because of its harmonic nature, indicating that an induced frequency can probably be determined by considering a certain regularization of the complex flexibility in a way to remove the discontinuities.

2 Model definition

The model considered in this paper is depicted in Fig. 1.

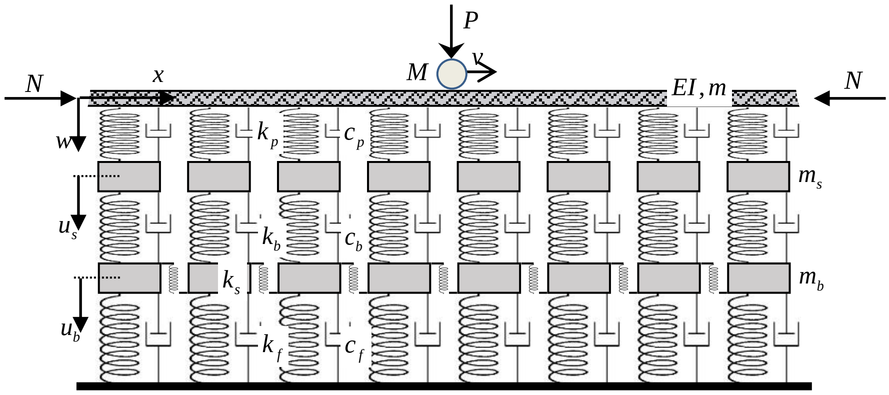

Figure 1. Three-layer model of the ballasted railway track.

The beam is characterized by its bending stiffness *EI* and mass per unit length *m*. Only the foundation layer characterized by its stiffness k_f and damping c_f is included in the one-layer model. In the two-layer model, in addition to the foundation layer, sleepers' mass m_s and rail pads' stiffness k_p and damping c_p are considered. In the three-layer model, as indicated in Fig. 1, ballast stiffness k_b, damping c_b and dynamically activated mass m_b are used in addition to the previous characteristics. Regarding the shear resistance of the model, in the one-layer model, the shear stiffness is presented by the Pasternak modulus, represented by the shear layer or rotational springs. In the three-layer model, shear stiffness is attributed to the ballast layer, k_s. In the two-layer version, there is no suitable component, therefore, shear springs are placed between the sleepers' masses. This may be an unusual approach, but since the track must exhibit some shear resistance, for modeling purposes this inclusion is considered adequate.

In addition, the beam might be subjected to an axial force *N* and is uniformly traversed by a moving mass *M* linked with constant force *P*, usually much higher than the associated mass's weight. The velocity is designated as *v* and unknown displacements as *w*, u_s and u_b, related to the beam, sleepers' layer, and ballast layer, respectively.

The governing equations for these three models in order of their complexity are given by

$$EIw_{,xxxx} + Nw_{,xx} - k_s w_{,xx} + mw_{,tt} + c_f w_{,t} + k_f w = p \tag{1}$$

$$\begin{aligned} &EIw_{,xxxx} + Nw_{,xx} + mw_{,tt} + c_p\left(w_{,t} - u_{s,t}\right) + k_p\left(w - u_s\right) = p \\ &m_s u_{s,tt} - k_s u_{s,xx} - c_p\left(w_{,t} - u_{s,t}\right) - k_p\left(w - u_s\right) + c_f u_{s,t} + k_f u_s = 0 \end{aligned} \tag{2}$$

$$\begin{aligned} &EIw_{,xxxx} + Nw_{,xx} + mw_{,tt} + c_p\left(w_{,t} - u_{s,t}\right) + k_p\left(w - u_s\right) = p \\ &m_s u_{s,tt} - c_p\left(w_{,t} - u_{s,t}\right) - k_p\left(w - u_s\right) + c_b\left(u_{s,t} - u_{b,t}\right) + k_b\left(u_s - u_b\right) = 0 \\ &m_b u_{b,tt} - k_s u_{b,xx} - c_b\left(u_{s,t} - u_{b,t}\right) - k_b\left(u_s - u_b\right) + c_f u_{b,t} + k_f u_b = 0 \end{aligned} \tag{3}$$

and the loading terms is

$$p\left(x,t\right) = \left(P - Mw_{0,tt}\left(t\right)\right)\delta\left(x - vt\right) \tag{4}$$

Here *x* is the fixed coordinate and *t* is the time.

During the solution of the problem defined by eq. (1)-(4), first fixed coordinates are replaced by moving ones, then dimensionless parameters are introduced, and Laplace and Fourier transforms are applied. The problem is solved analytically in the transformed domain, the inverse Fourier transform is still fully analytical, and only the inverse Laplace transform must be performed semianalytically. All these steps can be consulted in Dimitrovová [1, 3-6].

3 Complex flexibility and discontinuity lines

Complex flexibility, in other words K-function, of layered models is defined by

$$K(0,q)=\int_{-\infty}^{\infty}\frac{d_{Lr}}{d_L}\,\mathrm{d}p \tag{5}$$

where d_{Lr} and d_L are determinants of order L and L-1, with L being the number of layers. This means that for the three-layer model d_L is determinant of the third order of the matrix of coefficients for solution of the three unknown displacements in the transform space, and d_{Lr} is the second order determinant of the previous matrix reduced by eliminating the first column and row. For the one-layer model d_L is just a polynomial and d_{Lr} equals unity.

The definition is justified by the fact that for a constant moving force the steady-state deflection at the active point is given by

$$\tilde{w}(0,0)=\frac{4K(0,0)}{\pi} \tag{6}$$

This definition is supported by analogous definition of the equivalent complex stiffness introduced in Metrikine [7]. In previous eq. (5), p and q are transformed dimensionless moving coordinate ξ and time τ, representing just frequency, respectively. Dimensionless displacement in eq. (6) is ratio of the beam displacement and maximum static deflection exhibited by force P on the reference Winkler beam characterized by EI, m and k_f. For L=1, d_1 is given by

$$d_1=p^4-4p^2\left(\eta_N-\eta_S+\alpha^2\right)-4q^2+8\alpha pq+8\mathrm{i}\eta_f q-8\mathrm{i}\eta_f\alpha p+4 \tag{7}$$

where

$$\eta_N=\frac{N}{2\sqrt{k_f EI}}\,,\ \eta_s=\frac{k_s}{2\sqrt{k_f EI}}\,,\ \eta_f=\frac{c_f}{2\sqrt{mk_f}} \tag{8}$$

It was proven in Dimitrovová [4] that in this case, function K exhibits discontinuity in complex q-plane along two semi-infinite horizontal lines, spanning from plus or minus limiting frequency to plus or minus infinity at $\mathrm{i}\eta_f$ level. For the undamped case, and for the sake of simplicity for η_N=η_s=0, the limiting frequency equals unity for the critical velocity α=1, where α is the ratio of the velocity and critical velocity of the reference Winkler beam. This means that in the subcritical velocity range function K is always continues, because for its evaluation by contour integration p-roots for a given q are always complex. This means that the induced frequencies as roots of the characteristic equation (which is the same for all models)

$$2\eta_M q^2 K(0,q)=\pi \tag{9}$$

can be found and real roots q exist because K can be real-valued. In eq. (9)

$$\eta_M=\frac{M\chi}{m}\ \text{with}\ \chi=\sqrt[4]{\frac{k_f}{4EI}} \tag{10}$$

is the moving mass ratio.

In the other models the situation is much more complicated. Discontinuity exists anytime the nature of the p-roots changes. Discontinuity lines can thus be found by identification of places with real p-roots as a function of q. For specific α they can be plotted on the complex q-plane. For the identification, one can assume that $q=q_r+\mathrm{i}q_i$ and that q_r can be written as

$$q_{r_1,\alpha}=\alpha p+\Delta q_r\,,\ q_{r_2,\alpha}=\alpha p-\Delta q_r \tag{11}$$

When eq. (11) is introduced into d_L, L=2 or 3, then dependence on α is eliminated. Either the first or the second expression from eq. (11) can be used because d_L has the same form, except for the sign in its imaginary part. Then, by selecting some q_i, two equations have to be solved for two real unknowns, Δq_r and p. Knowing discontinuity lines for a specific α, with the help of eq. (11), they can be transformed to another α-value without the need of solving the equations again.

The important conclusion of this analysis is that the discontinuity lines occur always in the half complex q-plane with positive imaginary part, i.e., in the stable region. Therefore, this property does not interfere with detection of unstable cases. However, it affects the semianalytical closed form solution, namely the unsteady harmonic part. This part corresponds to a sum of residues at induced frequencies' values, therefore, when they do not exist, the corresponding residue cannot be evaluated.

4 Regularization technique

It is of note that even if the frequency lines can suddenly end due to some discontinuity in *K*-function as described in previous section, regarding the time series of the active point or evolution of full deflection shapes, there is no sign of some sudden change in vibration behavior. Lack of some induced frequency is compensated for by the truly transient part of the solution obtained by integrating along the branch cuts. Nevertheless, due to such a smooth passage, it seems convenient to introduce some kind of regularization to overcome the discontinuity and smoothen the *K*-function.

The proposed technique is simple. Any time a value of *K*-function is needed for a given frequency, the evaluation starts in unstable region by using the same real part of this frequency and some small negative imaginary part. Then, by smooth change of the imaginary part of the selected q until the final value and by tracking the roots for the contour integral as the closest ones to the previous ones, the desired value is obtained.

In the unstable region, there are always only two p-roots with a positive imaginary part, needed for the contour integration along the half circumference in the upper half plane. These two roots are followed by successively selecting the two nearest roots while the imaginary part of the selected frequency reaches its final value. This will lead to a smooth change of the *K*-function without discontinuity.

4.1 Two-layer model

By analyzing the two-layer model in the subcritical velocity range and without damping, it is found that the discontinuity in the *K*- function limits finding the real induced frequencies. Nevertheless, the level of discontinuity is almost of the order of the required precision, meaning that when it is needed to have real-valued *K*-function, then its imaginary part is non-zero, but almost negligible. Therefore, it can be neglected, and no regularization is necessary in most cases. This was useful in analyzing the dynamic interaction between proximate masses in Dimitrovová [6]. Several examples show that the agreement is excellent, and no limitation was imposed by the discontinuity in the *K*-function.

4.2 Three-layer model

In the three-layer model, the situation is not so simple. The discontinuity in the *K*-function is already so strong that it can be neglected only for very low α. In the undamped case, the *K*-function is never real-valued. This means that there cannot exist real induced frequencies. One would expect that in subcritical velocity range and without damping, the unsteady harmonic part of the solution would have a constant amplitude as in the one-layer model. But that is not true. The induced frequency already must have an imaginary part that makes the time series look damped.

In order to show some examples, other parameters must be introduced. To identify the three-layer model in a dimensionless way, the reference Winkler beam is used. All parameters from Fig. 1 are assumed as distributed. Then the model is defined by stiffness and mass ratios by

$$\mu_s = \frac{m_s}{m}\ ,\ \mu_b = \frac{m_b}{m}\ ,\ \kappa_p = \frac{k_p}{k_f}\ ,\ \kappa_b = \frac{k_b}{k_f} \tag{12}$$

and damping ratios

$$\eta_p = \frac{c_p}{2\sqrt{mk_f}}\;,\;\eta_b = \frac{c_b}{2\sqrt{mk_f}}\;,\;\eta_f = \frac{c_f}{2\sqrt{mk_f}} \tag{13}$$

For instance, for a case with μ_s=3, μ_b=10, κ_p=0.03, κ_b=0.1, η_s=0, all damping ratios with negligible value of $1\cdot10^{-10}$ and moving mass ratio of 20, induced frequencies obtained by the regularization technique proposed in this paper are: for α=0.05 it is $0.0678+i1.4425\cdot10^{-7}$; for α=0.1 it is $0.06557+i1.5653\cdot10^{-4}$ and for α=0.15 it is $0.06097+i2.3540\cdot10^{-4}$.

Fig. 2 shows a comparison of two solutions indicating the displacement of the active point as a function of time: the complete one obtained by numerical integration and obtained by a semianalytical approach and the newly determined induced frequencies. These results were also validated on a representative finite beam, similarly as in Dimitrovová [1, 3 -6]. It can be concluded from Fig. 2 that the agreement is excellent.

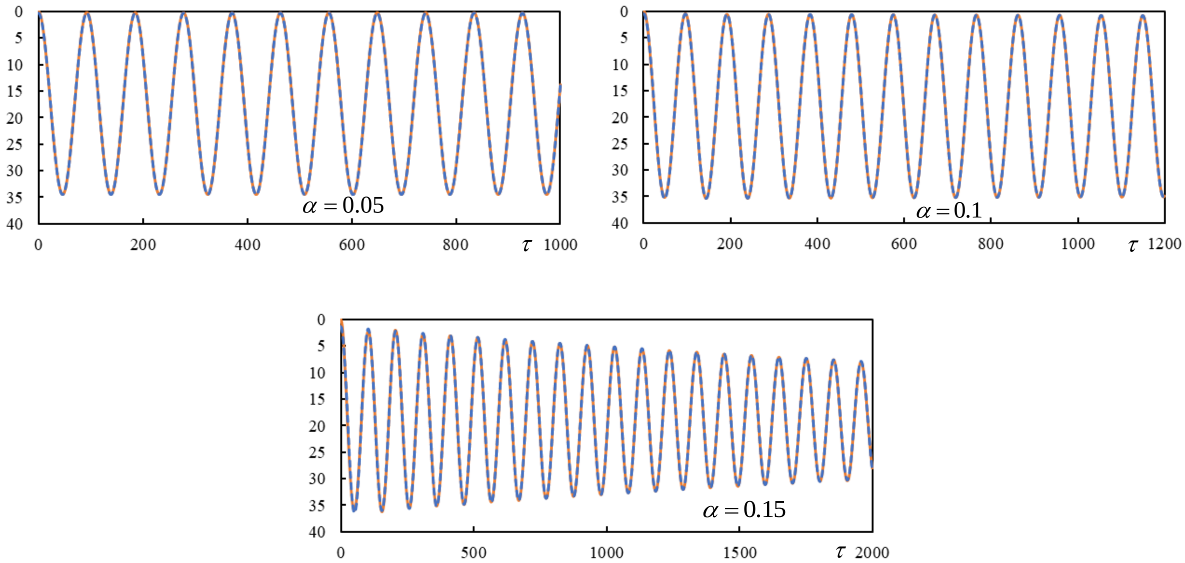

Figure 2. Time series of the active point as a function of the dimensionless time for α=0.05; 0.1 and 0.15.

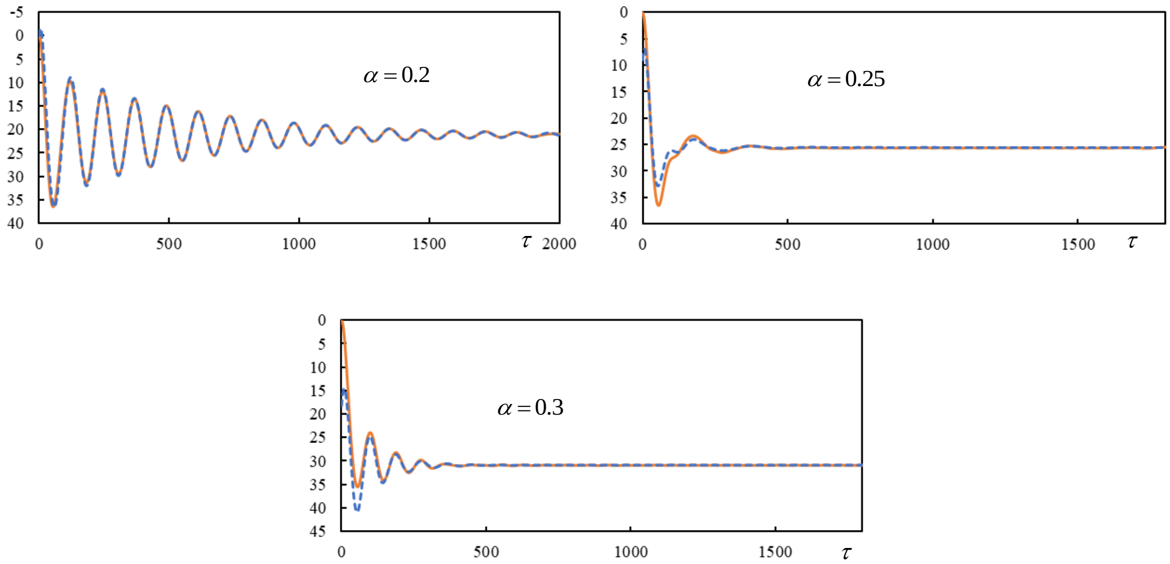

Figure 3. Time series of the active point as a function of the dimensionless time for α=0.2; 0.25 and 0.3.

In Fig.3 higher velocities are presented. α=0.2 with $0.05124+i0.00184$ and $0.7474+i0.03465$; α=0.25 with

0.03126+i0.00799 and 0.07367+i0.02493 and α=0.3 with 0.07081+i0.01101. Some differences are already noticeable. These are not related to the accuracy of the newly determined induced frequencies, but generally, to the adequacy of approximating the full solution by its harmonic part. Therefore, the discrepancies are related to the fact that the harmonic solution is not fulfilling the initial condition, as it is clearly seen from the difference between the displacement values at τ=0. It is of note that for these problem's parameters in the absence of damping, 3 well-formed critical velocities of a moving force can be determined. The lowest value is 0.437, thus, all cases in Figs. 2 and 3 are in the subcritical velocity range. The damping is negligible, but the response in some cases looks like highly damped. In addition, this damping effect is more pronounced for α=0.25 than for α=0.3, i.e., it is not more significant when closer to the critical velocity. This is different from the other two models, as already mentioned.

Even if the regularization technique was presented without mathematical proof, it was demonstrated that the results are promising.

5 Conclusions

In this paper, a new regularization technique was introduced to remove discontinuities in the K-function and thus allow a better analysis of undamped cases where the dynamic interaction between proximate masses may be important. The technique follows a simple rule of smoothly changing the positions of the roots identified by the closest roots to the previous ones. The idea is based on the fact that interruption of frequency lines, that is, sudden switch to a situation where the induced frequencies do not exist, does not translate into a sudden abrupt change in the time series. Conversely, a smooth variation of the velocity in the subcritical range of velocities always leads to a smooth variation in the time series, regardless of whether induced frequencies exist or not.

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