Investigation of Band Structures for Different Lattice Types in Sierpinski Phononic Fractal Crystals with Local Resonance

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Abstract. This article presents an analysis of the band properties and forced response in phononic crystals that use fractal structures, focusing on square and triangular Brillouin lattices. Initially, we describe the construction of phononic crystals based on fractals, using fractal patterns such as the Sierpinski set or quasi-Sierpinski. These fractal structures allow the creation of complex and branched arrangements at different scales, resulting in different properties. Different types of Brillouin networks will be used to analyze and verify the influences on the band structures of phononic crystals. Furthermore, we examine the forced response of phononic crystals, analyzing how these structures react when subjected to external excitations. Using the finite element method, we obtained the forced responses, revealing the resonance modes and attenuation characteristics at different frequencies. This study highlights the importance of fractal structures in manipulating the acoustic properties of phononic crystals. The analysis of the band structures and forced response offers valuable insights for the development of phononic devices with applications in acoustic insulation, frequency selective transmission and other related areas. These results contribute to the understanding and advancement of fractal phononic crystals, paving the way for future research and promising technological applications, both in terms of frequency and attenuation, depending on the complexity and hierarchy of the fractal structures used.

Keywords: Sierpinski fractal, hierarchic, periodic structures.

1 Introduction

The exploration of wave propagation within heterogeneous elastic and acoustic mediums has garnered increasing attention in recent years. This rise in interest can largely be attributed to the surging demand for products and construction methodologies aimed at enhancing acoustic comfort. These endeavors encompass minimizing noise-related disturbances and mitigating external vibrations within sensitive structures [1–3].

Periodic structures, characterized by the repetitive arrangement of a fundamental unit, possess the remarkable capability to manipulate wave behavior. Their extensive application spans various domains, encompassing mechanical systems employed for tasks like vibration reduction, concealment, and imaging. Phononic crystals (PnCs), representative examples of periodic structures, exhibit the capacity to induce frequency-specific gaps known as band gaps. These gaps emerge through the destructive interference of waves. They effectively hinder the propagation of free waves and materialize due to impedance mismatches achieved through spatial modulation within monophase materials or via the amalgamation of materials with disparate elastic properties [4–6].

Distinct from band gaps stemming from Bragg scattering, band gaps arising from local resonance within PnCs depend directly on the symmetrical and periodic traits of the structure. The fusion of PnCs with the incorporation of locally resonant elements leads to the creation of Locally Resonant Phononic Crystals (LRPnCs), often referred to as mechanical metamaterials. These entities typically feature a compliant polymer with a stiffness orders of magnitude lower than typical materials, combined with stiffer substances. This composite design yields low-frequency gaps, suggesting applications like vibration reduction at low frequencies and sound insulation.

Recent contributions in this field include the evaluation of localized resonators in meta-concrete structures by Miranda et al. [1]. Additionally, Yip and John presented an innovative material composition for LRPnCs, involving a dense core (carbon) enveloped by a polymer layer (silicone rubber), followed by a carbon layer. Both studies incorporated Plane Wave Expansion (PWE) techniques or related methodologies for the modeling and analysis of these structures.

In summary, the investigation of wave propagation in heterogeneous elastic and acoustic mediums has grown in prominence, driven by the demand for enhanced acoustic comfort. Periodic structures like PnCs exhibit remarkable abilities in wave manipulation, while the fusion of PnCs with local resonance leads to the creation of LRPnCs with diverse applications. Recent works by various researchers have provided valuable insights into this realm, employing methods like PWE for structural analysis.

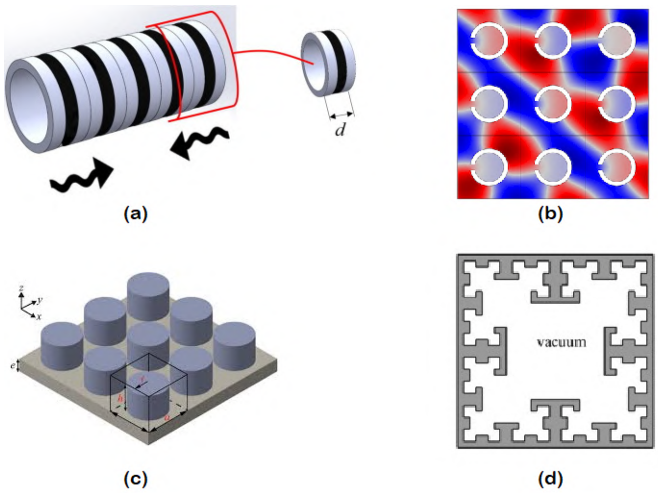


Figure 1. (a) - Jin et al. [7] study periodic cylinders for wave attenuation, Elford et al. [8] development a numerical modeling of sonic crystals with resonant array elements, and Jing-Fu et al. [9] evaluated T-square fractal shape holes on PnCs.

Metamaterials represent artificial substances intentionally engineered to possess attributes absent in natural materials. They consist of repetitive structures at a scale smaller than the wavelength of the targeted waves for manipulation. Metamaterials serve to regulate the propagation of electromagnetic, acoustic, or various wave types. For instance, they can yield materials with a negative refractive index, causing light to deviate contrary to that in natural substances.

The impact of fractal-shaped inclusions on PnC band structures has garnered growing attention, unveiling novel phenomena and behaviors. The influence of periodic fractal inclusions on the frequency response of 2-D PnCs was explored. Kuo and Piazza [10] introduced a T-square fractal design for microscale phononic bandgap structures in aluminum media and explored fractal PnCs. Prior research mainly concentrated on fractal hollow and solid-air phononic configurations. Inspired by these concepts, this study investigates band gaps within a solid-solid fractal phononic structure, presenting a marked departure from preceding investigations. Li et al. [11] analyzed the effect of T-square fractal shape holes on the 2-D PnC band structure, noting the influence of increased fractal levels on band gaps.

This research assesses wave propagation through an isotropic thin plate, modeled on solid theory, and composed of a 2-D array of structural units based on The Sierpinski carpet fractal is a well-known and visually intriguing geometric pattern that falls under the category of fractals. It is named after the Polish mathematician Wacław Sierpiński, who made significant contributions to the field of mathematics, including the study of fractals. The Sierpinski carpet fractal is created through a recursive process of dividing a square into smaller squares and removing a central square from each iteration. The remaining squares are then subject to the same process, resulting in an infinite sequence of iterations that generates the characteristic pattern.

Starting with a solid square, the first iteration involves removing a smaller square from its center. In each subsequent iteration, the remaining squares are divided into nine equal smaller squares, and the central one is removed. This process is repeated indefinitely, leading to the emergence of a complex, self-replicating pattern with a carpet-like appearance. The Sierpinski carpet exhibits self-similarity, meaning that as you zoom in on any part of the fractal, you will find patterns similar to the entire structure. Despite its intricate design, the Sierpinski carpet is relatively simple to describe using recursive algorithms, making it a popular subject for studying fractals and their properties.

2 Numerical Methods

2.1 Plane Wave Expansion

The band structure referring to the behavior of waves propagating in the phononic crystal is determined by two methods: Plane Wave Expansion - PWE and by Finite Elements - from COMSOL. Firstly, the PWE formulation for a 2D PnC thin plate [1] is used to determine the band structure of wave propagation restricted to the plane $(x-y)$, solving the eigenvalue problem only for the First Irreducible Brillouin Zone (FIBZ). The govern differential equation is given by Sigalas and Economou [2]:

$$-\alpha\frac{\partial^2 w}{\partial t^2}=\frac{\partial^2}{\partial x^2}\left(D\frac{\partial^2 w}{\partial x^2}+\beta\frac{\partial^2 w}{\partial y^2}\right)+2\frac{\partial^2}{\partial x\partial y}\left(\gamma\frac{\partial^2 w}{\partial x\partial y}\right)+\frac{\partial^2}{\partial y^2}\left(D\frac{\partial^2 w}{\partial y^2}+\beta\frac{\partial^2 w}{\partial x^2}\right), \tag{1}$$

where w is the transverse displacement, $D=\frac{Eh^3}{12(1-\nu^2)}$ is the flexural rigidity with E being the Young's modulus and ν is the Poisson's ratio, $\alpha=\rho h$, $\beta=D\nu$, and $\gamma=D(1-\nu)$ are periodic functions of the position vector $\mathbf{r}(x,y)$. Applying the Floquet-Bloch's theorem, the w can be expressed as $w(\mathbf{r},t)=e^{j(\mathbf{k}\cdot\mathbf{r}-\omega t)}w_{\mathbf{k}}(r)$. With $\mathbf{k}=(k_x,k_y)$, the Bloch wave vector, ω, angular frequency, t, time, and $j=\sqrt{-1}$. $w_{\mathbf{k}}$ has periodicity equivalent to the periodicity of the phononic crystal and is expanded in Fourier series as:

$$w_{\mathbf{k}}=\sum_{\mathbf{g}_1}e^{j\mathbf{g}_1\cdot\mathbf{r}}A_{\mathbf{g}_1}, \tag{2}$$

$\mathbf{g}_1=\frac{2\pi}{a}(n_1,n_2)$ is 2D reciprocal lattice vector for square lattice with n_1 and n_2 = $[0,\pm1,\pm2,...,\pm n]$, a is the lattice parameter and the Fourier coefficient, $A_{\mathbf{g}_1}$. The periodic functions, $\alpha(\mathbf{r})$, $\beta(\mathbf{r})$, $D(\mathbf{r})$ and $\gamma(\mathbf{r})$, are expanded in Fourier series as:

$$H(\mathbf{r})=\sum_{\mathbf{g}_2}e^{j\mathbf{g}_2\cdot\mathbf{r}}H_{\mathbf{g}_2}, \tag{3}$$

$H(\mathbf{r})$ is one of the plate parameters. The corresponding Fourier coefficient is given by:

$$H_{\mathbf{g}_2}=\begin{cases} fH_A+(1-f)H_B & for \quad \mathbf{g}_2=\mathbf{0} \\ (H_A-H_B)F_{\mathbf{g}_2} & for \quad \mathbf{g}_2\neq\mathbf{0}\end{cases}, \tag{4}$$

with $\mathbf{g}_2=\frac{2\pi}{a}(\bar{n}_1,\bar{n}_2)$, $\bar{n}_1$ and $\bar{n}_2$ = $0,\pm1,\pm2,...,\pm n$, $H_{\mathbf{g}_2}$ refers to a Toeplitz matrix, f is defined as the ratio between the cross sectional area of a cylinder and primitive unit cell, i.e., filling fraction of inclusion, and the structure function, $F_{\mathbf{g}_2}$, is defined as:

$$F_{\mathbf{g}_2}=\frac{1}{S}\int_S e^{-j\mathbf{g}\cdot\mathbf{r}}dr^2, \tag{5}$$

where S refers to the area of the unit cell and the integral operator being calculated over the cross section of the host material.

1. In cases where the PnC consists of circular cylinder inclusions (radius r_o), the filling fraction is given by:

$$f=\frac{\pi r_o^2}{a^2} \quad \text{and} \quad F_{\mathbf{g}_2}=\frac{2f}{|\mathbf{g}_2|r_o}J_1(|\mathbf{g}_2|r_o), \tag{6}$$

   where $J_1(|\mathbf{g}_2|r_o)$ is the Bessel function of the first kind.

2. For square section of width $2l$

$$f=\frac{4l^2}{a^2} \quad \text{and} \quad F_{\mathbf{g}_2}=f\frac{\sin(g_1 l)}{(g_1 l)}\frac{\sin(g_2 l)}{(g_2 l)}. \tag{7}$$

Substituting the considerations of displacement fields of flexural waves and the Eq. (3) in Eq. (1), we can express the following eigenvalue problem:

$$\begin{aligned}\omega^2\sum_{\mathbf{g}_1}[\alpha]_{\mathbf{g}_2}A_{\mathbf{k}+\mathbf{g}_1} &= \sum_{\mathbf{g}_1}(\mathbf{k}+\mathbf{g}_1)_x^2(\mathbf{k}+\mathbf{g}_3)_x^2\,[D]_{\mathbf{g}_2}A_{\mathbf{k}+\mathbf{g}_1}+\\ &+ \sum_{\mathbf{g}_1}(\mathbf{k}+\mathbf{g}_1)_y^2(\mathbf{k}+\mathbf{g}_3)_x^2\,[\beta]_{\mathbf{g}_2}A_{\mathbf{k}+\mathbf{g}_1}+\\ &+ 2\sum_{\mathbf{g}_1}(\mathbf{k}+\mathbf{g}_1)_x(\mathbf{k}+\mathbf{g}_1)_y(\mathbf{k}+\mathbf{g}_3)_x(\mathbf{k}+\mathbf{g}_3)_y\,[\gamma]_{\mathbf{g}_2}A_{\mathbf{k}+\mathbf{g}_1}+\\ &+ \sum_{\mathbf{g}_1}(\mathbf{k}+\mathbf{g}_1)_y^2(\mathbf{k}+\mathbf{g}_3)_y^2\,[D]_{G_2}A_{\mathbf{k}+\mathbf{g}_1}+\\ &+ \sum_{\mathbf{g}_1}(\mathbf{k}+\mathbf{g}_1)_x^2(\mathbf{k}+\mathbf{g}_3)_y^2\,[\beta]_{\mathbf{g}_2}A_{\mathbf{k}+\mathbf{g}_1},\end{aligned} \tag{8}$$

$\mathbf{g}_3 = \mathbf{g}_1 + \mathbf{g}_2$. Therefore, rewriting the Eq. (8) in matrix form:

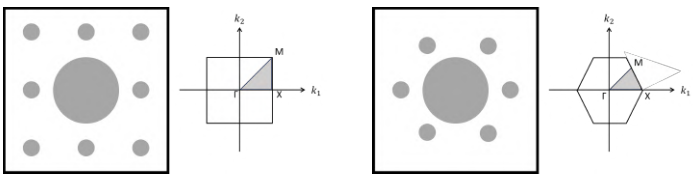
$$\boldsymbol{\omega^2}\mathbf{P}\boldsymbol{A_{k+g_1}} = \mathbf{Q}\boldsymbol{A_{k+g_1}}. \tag{9}$$

Solving the Eq. (9), eigenfrequency is found, ω, for each real value of Bloch vector **k** within FIBZ [3].

2.2 Fractals and Brillouin zones

The arrangement of the locking elements follows a square grid structure, Fig. 2 (a). The first irreducible Brillouin zone, which is a region in wave space, is represented by a square geometric figure. These phononic crystals are designed to control and modulate the properties of sound waves at various frequencies, creating forbidden frequency bands known as band gaps. Square lattice structures are often used in applications involving sound insulation systems, frequency selective filters, and advanced phononic devices Maurin et al. [12].

In this case, Fig. 2 (b) , the arrangement of blocking elements follows a triangular network structure. The first irreducible Brillouin zone is represented by a hexagon in wave space. Phononic crystals with a triangular lattice configuration have distinct acoustic properties from those with a square lattice. They are often exploited in contexts where triangular symmetry is important, such as in hexagonal materials and crystalline systems that exhibit this symmetry. These crystals can be used to control the propagation of sound waves and create band gaps at specific frequencies, being applied in areas such as material acoustics, acoustic waveguides and phononic filtering devices [12].

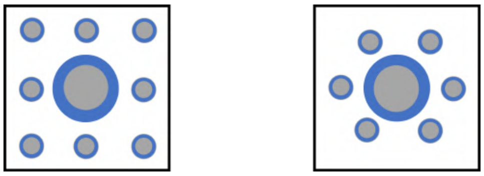


(a) (b)

Figure 2. Brillouin zones in two-dimensional for (a) - square and (b) - triangular lattices, each with their respective unit cell for fractal distribution.

2.3 Local Resonance

Local resonance in phononic crystals refers to a phenomenon where certain elements or inclusions within the structure of the phononic crystal begin to vibrate or oscillate strongly in response to the incidence of acoustic waves at a specific frequency. These inclusions are designed to be tuned to the frequency of the sound waves you want to control or block. When inclusions enter local resonance, they absorb a significant amount of energy from acoustic waves, which can lead to selective suppression or attenuation of these frequencies. This results in the formation of forbidden bands, also known as band gaps, where the propagation of acoustic waves at certain frequencies is impeded or greatly reduced. Local resonance in phononic crystals is a powerful strategy for designing materials with customized acoustic properties. By carefully tuning the geometric and mechanical characteristics of inclusions, it is possible to create band gaps in specific frequency ranges. This has practical applications in many areas such as sound insulation, vibration control, sound focusing devices and frequency selective filtering. In this work, we also sought to evaluate the effect of local resonance on PnC. Thus, as seen in Fig. 3 (a) and (b), a soft material (silicone) was added around the core (solid steel).



(a) (b)

Figure 3. Unit cell with uncoated core - two materials (a) and coated - three materials (b).

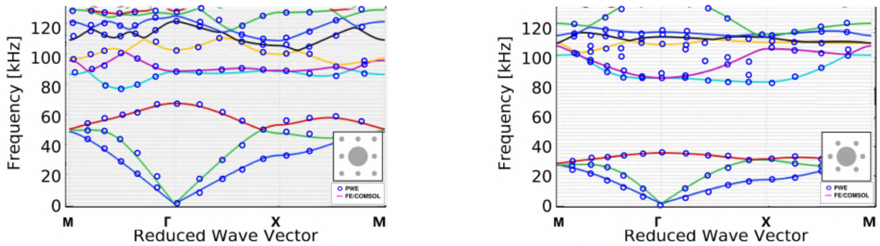
Table 2.3 presents the properties of the materials used in this work.

Table 1. Material properties [13]

Color	Material	Density [kg/m^3]	Young's Modulus [GPa]
Gray	Epoxy	2,500	40
Blue	Silicone	1,300	1.175
Black	Steel	7,875	210

3 Simulated Results

The preliminary results presented in this work were validated and proved to be concise (Fig. 4 (a) - (b)), when compared with those developed in Miranda et al. [1] and Dos Santos et al. [14] Using the case of Fig. 2 (a) - (b), the dispersion curves for the squad lattice PnC, considering the case of evaluation of the first and second-stage quasi-Sierpinski fractal.



(a) (b)

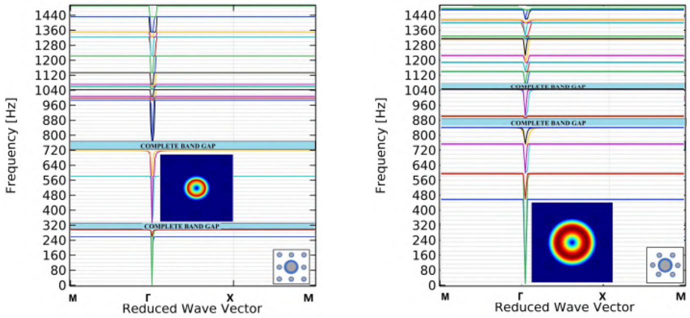
Figure 4. Comparison of band structures found for PWE and FE/COMSOL (a) squad and (b) triangular lattices.

LRPnCs with local resonance caused by the silicone ring around the solid steel core. One can observe the presence of two complete bandgaps, these located in the frequencies between 300 and 330 Hz, and from 720 to 780 Hz. In this initial analysis, the presence of valleys that demonstrate the resonance provoked in the system, this one directed to the Γ region of FIBZ, Fig. 5 (a).

For PnC with fractal distribution in its second stage with application of local resonance inclusions (Fig. 5 (b)), where there is triangular lattice it is possible to clearly verify the behavior of the resonators and the position within the FIBZ was modified. Here, perhaps due to the reduction in the number of inclusions, from eight (8) to six (6), which also implied mass reduction, the bandgap frequencies were changed to a higher range, where the first bandgap was 850 to 900 Hz, and the second from 1050 to 1110 Hz. The valley caused by the local resonance remained in the region Γ, which is equivalent to $k = 1$.

3.1 Final Remarks

In this study, the band structure of PnC with two types of Brillouin lattices based on Sierpinski mat were investigated. In addition, the phenomenon of local resonance was added through a steel core surrounded by a



(a) (b)

Figure 5. Unit cell with uncoated core - two materials (a) and coated - three materials (b).

silicone ring, both embedded in an epoxy matrix. Which were investigated based on finite element simulations and compared with the PWE, for the first stage of the fractal. We can summarize the results as follows:

- Square lattice PnC has bandgaps at lower frequencies, while squad lattice PnC has bands at higher frequencies;
- Although both networks presented the same amount of bandgaps for the evaluated frequency range, PnC with a triangular network presented narrower bandgaps;
- It is intended to extend the formulation developed in this work to evaluate the behavior of evanescent waves, through the EPWE method (extended plane wave expansion method).

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