An algorithm to compute the parameters of a Generalized Kelvin Chain model to represent aging creep of concrete

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Abstract. The numerical modelling of viscoelastic behavior of large concrete structures can be optimized by representing the compliance function in terms of series of exponentials, usually termed Dirichlet series, Prony series, or Generalized Kelvin Chain mechanical model. The viscoelastic behavior of concrete is usually described in terms of a compliance function obtained from classical creep tests or design codes, being, thus, necessary to devise a workflow to derive the set of retardation times and moduli that completely describe the correspondent Generalized Kelvin Chain model. Direct fitting of the Generalized Kelvin Chain model to a given compliance function is usually not feasible, as the problem becomes ill-posed if the chain contains a reasonable number of units, which is necessary for correct representing the viscoelastic behavior. The present work presents an open-source Python algorithm to compute the parameters of a Generalized Kelvin Chain model associated to a set of aging compliance functions. The use of the algorithm is exemplified by computing the Generalized Kelvin Chain model associated to compliance functions obtained from Eurocode 2 and *fib* Model Code 2010, which are common viscoelastic models used in engineering practice.

Keywords: viscoelasticity, creep, early ages, Generalized Kelvin Chain model, concrete

1 Introduction

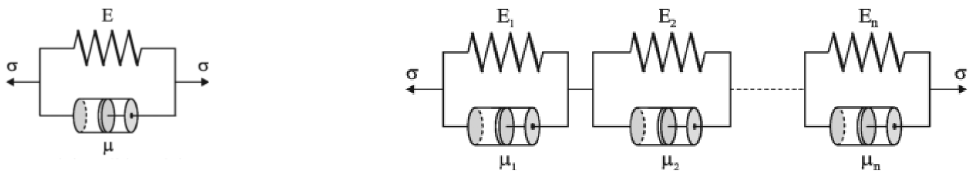
Consideration of delayed deformation caused by creep is essential in serviceability checks of deformation-sensitive concrete structures, such as reactor vessels, shells, and large span bridges, whose functionality and safety can be critically affected if unpredicted displacements start to develop due to the viscoelastic behavior of concrete [1]. Efficient numerical modelling of viscoelastic behavior of large concrete structures, now routinely performed with Finite Element Method software, demands that the viscoelastic constitutive equation be written in a differential rather than in an integral form [2], [3]. This can be done by writing the compliance function not as a continuous single function, but as a series of exponentials, in what is usually referred to as Dirichlet series, Prony series, or, to emphasize its relation to the Kelvin viscoelastic mechanical model, the Generalized Kelvin Chain model [1]. In practical problems involving concrete structures, this approach involves going from the compliance function, which is the experimental property obtained in classical creep tests or provided by design codes, to a Generalized Kelvin Chain model, which is completely defined by the set of retardation times and modulus of all its Kelvin units [2]–[4]. Direct fitting of the Generalized Kelvin Chain model to compliance functions, if not performed with proper precautions, is usually not feasible, as the problem becomes ill-posed if the chain contains a reasonable number of units, which is however necessary for correct representing the viscoelastic behavior. This matter has been addressed by some previous publications, such as [1], [2], [5].

This work had the object to develop an open-source algorithm to generate the coefficients of a Generalized Kelvin Chain model associated to a given set of aging compliance functions. A feature for data presmoothing was also implemented, to aid when dealing with noisy or missing point data in compliance functions, which is a typical situation in experimental data. Also, algorithms to compute compliances accordingly to Eurocode 2 [6] and *fib* Model Code 2010 [7] design codes were implemented to provide means to generate data for validation of the fitting algorithm.

2 Background theory

2.1 Generalized Kelvin Chain model

The Generalized Kelvin Chain model can be intuitively understood on the basis of simple viscoelastic mechanical models. Usually, viscoelastic mechanical models are composed by combinations of two basic mechanical elements: the spring and the damper, characterized, respectively, by an elastic constant E and a viscosity μ [8]. Different arrangements of these basic mechanical elements (*e.g.*, in series, in parallel, and a mix of both) give rise to models capable of representing a wide range of behaviors. When a spring and a damper are arranged in a parallel fashion, the resultant model is called a Kelvin or a Kelvin-Voigt model (Figure 1a), and the arrangement in series of several Kelvin models or units, each with its own characteristic properties (E_i, μ_i), leads to the Generalized Kelvin Chain model (Figure 1b).



(a) (b)

Figure 1 Viscoelastic models: (a) Kelvin model; (b) Generalized Kelvin Chain model. In the models, E stands for elastic constant and μ is the viscosity. Adapted from [8].

Because each Kelvin unit may have different individual properties, the Generalized Kelvin Chain model presents a great flexibility in representing different viscoelastic behaviors and, thus, is widely used in the literature for viscoelastic behavior modeling [2]–[4], [9]. Also, in the context of concrete creep, in which aging takes an important role, a Generalized Kelvin Chain model allows for each Kelvin model to possess a different aging kinetic, in which case the elastic constant E and the viscosity μ are not constants but may be dependent on the age of the material. This helps the model to better adhere to real concrete creep data more easily. The mathematical representation of the lumped compliance of an aging Generalized Kelvin Chain model, composed by n aging Kelvin unit, is given by:

$$J(t,t') = \sum_{i=1}^{n} \left[\frac{1}{E_i(t')} \left(1 - e^{-(t-t')/\tau_i} \right) \right] \tag{1}$$

In which: the two independent variables t and t', required to represent the aging viscoelastic problem and computed using the same time reference (*i.e.*, if $t = 0$, then $t' = 0$), represents, respectively, the time along which the viscoelastic phenomenon will develop, and the time in which the loading was applied to the material; $J(t,t')$ is the creep compliance, which is the material property of interest in creep problems, *i.e.*, those problems represented by the strain-stress equation $\varepsilon(t) = \sigma \cdot J(t,t')$, with σ being the stress applied to the material in the time t' and $\varepsilon(t)$ the consequent viscoelastic strain developed in time t; the coefficients $E_i(t')$ are the elastic coefficients of the aging Kelvin models and, thus, depend on t'; τ_i is a property called retardation time, introduced to ease the mathematical notation of the problem, given, for the i-th Kelvin model, by $\tau_i = \mu_i(t')/E_i(t')$ [2].

The retardation time holds an interesting physical meaning: it is the time taken by a Kelvin model to achieve a strain equal to $1/e$ of the final strain in a creep problem. In other words, it conveys the information of how fast or slow the viscoelastic effects of that Kelvin unit develop in time, with small retardation times representing faster processes in time. Sometimes the Generalized Kelvin Chain model is represented having a Kelvin model with just a spring term, in order to represent the instantaneous response of the material when subjected to load (*i.e.*, the so-called "elastic" strain). However, such approach is not mandatory, and the more general form presented in Figure 1b and eq. (1) can be maintained if one recalls that an instantaneous response could still be represented by a very small retardation time, such as that $\tau_i \to 0$ [2].

Equation (1) can be also referred as Prony series, if the spacing between consecutive τ_i is constant, and pertains to a more general class of series representation based on exponentials called Dirichlet series [1], [10].

Such recognition opens the possibility to explore the subject from the perspective of other fields, such as mathematics and polymer science [4], [9]–[13].

2.2 Fitting a Generalized Kelvin Chain model to real data

In practical numerical analysis of concrete structures accounting creep and aging, the model given in eq. (1) is the usual choice for the stress-strain constitutive equation [1], [2]. The construction of such model for a particular material, however, is not a simple task. The standard strategy could be to simply attempt a direct fitting of eq. (1) to the available viscoelastic data of the material. This data may come from experimental compliance curves obtained in creep tests or compliances obtained from a particular design model (such as design codes such as Eurocode [6] or *fib* Model Code [7], or theoretical models such as the B3 [14] or B4 models [15], which allow computing compliance curves from basic information of the material – concrete class, geometric information of the structural member analyzed, etc.). With the compliance data at hand, one would then determine the desired number of Kelvin elements (i.e., define the value of n in eq. (1)), and use any non-linear curve fitting algorithm to obtain the values of E_i and τ_i that would minimize the difference between the compliances values predicted by eq. (1) and those from the available compliance data. If aging is involved, as in the context of this work, the procedure is the same, but it would be repeated for every loading age *t'* of interest.

The direct fitting approach, however, is long known to lead to an ill-posed problem, in which very different set of values for E_i and τ_i lead to considerably similar results, are over-sensitive to small alterations of the data, lead to no convergence at all in optimization algorithms used for fitting, or may result in some negative moduli, for some Kelvin units, which although not mathematical incorrect should be impossible from the physical meaning of the problem [2], [16]. Finding the proper values of the parameters of the model in eq. (1) becomes more amenable if some recommendations found in the literature are followed [1], [2], [13], [17]:

- Instead of seeking for E_i and τ_i, one can specify beforehand the values of τ_i and perform the fitting of eq. (1) over experimental data only to find E_i, which makes the fitting problem more amenable from the optimization point of view [1], [16], [17];
- The choice of τ_i is bounded: a lower bound is necessary so excessive close τ_i's do not lead to loss of uniqueness of fit, and an upper bound is necessary to ensure the τ_i's cover the range of retardation times contained in the creep data [1], [2], [17];
- The first retardation time, τ_1, should be taken as a very small value, such as $\tau_1 = 10^{-9}$ days, so the immediate response of the material can be properly modelled [2];
- The second smallest retardation time, τ_2, should be taken as smaller than three times the smallest time delay after load application, τ_{min}, for which the viscoelastic response is of interest (*i.e.*, $\tau_2 \leq 3\tau_{min}$) [2];
- The largest retardation time, τ_n, should be taken as larger than half of the largest time delay after load application, τ_{max}, for which the viscoelastic response is of interest (*i.e.*, $\tau_n \geq 0.5\tau_{max}$) [2];
- It is usual to uniformly distribute the retardation times in the logarithmic scale, such as that [2], [13]:

$$\tau_i = 10^{i-2}\tau_2 \qquad (i = 3,4, \dots, N) \tag{2}$$

- When fitting aging compliance functions, the loading ages and data points on each compliance curve should be sampled uniformly in the logarithmic scale [1].

In addition to these recommendations, the application of bounds in the optimization problem when performing the curve fitting allows to guarantee the existence of no negative moduli. Regarding such matter, additional recommendations to improve adequacy and physical meaning of eq. (1) to concrete creep data exist, such as using a slightly different forms for eq. (1) and (2) with the use of the reduced time concept [2]. Also, completely different approaches than outlined here also provide further assurance towards the physical meaning of the Generalized Kelvin Chain parameters, *e.g.*, computation of a continuous retardation spectrum and its subsequent discretization in order to build the model in eq. (1) [18]–[20].

2.3 Creep laws for data presmoothing

When one is working with experimental data, imprecision, scattering or loss of data, which are inherent to experimental measurements, may affect the quality of a direct fitting of eq. (1). This may introduce undesirable features in the resultant fit, such as wavy oscillations or negative modulus in some Kelvin units [2], [5]. To mitigate such effects, the compliance data can be previously smoothed by fitting a function of choice, which is then used to generate data points for fitting eq. (1).

The literature on concrete creep have suggested several aging creep laws that may be used for presmoothing. Bažant and Osman [21] provide a collection of some of the best-known creep formulas at the time of their publication. For the purpose of exemplifying some relevant concepts for the scope of this work, the double-power law is here introduced, being given by [21]:

$$J(t,t') = \frac{1}{E_0} + \frac{\varphi_1}{E_0} t'^{-m} (t - t')^n \tag{3}$$

In which t and t' conserve the same meaning as given in eq. (1), and E_0, φ_1, m, and n are the four parameters that determine the law. This law is known to reasonably adhere to data of concrete creep at constant temperature and water content (the so-called basic creep) but may provide excessive final slopes when fitted to long-term tests, for which the double-power logarithmic law seems to be more adequate [22].

At first inspection, one may observe that eq. (2) may present limitations to properly model the aging dependence of the instantaneous response of creep, since the first term $1/E_0$ is age independent and, thus, all aging compliances would have the same value for $t = t' = 0$. To improve this aspect, nothing prevents that a modified version of this law is used, by introducing an age dependency in the first term with a three-parameter model [23], leading to:

$$J(t,t') = \frac{1}{E_0 \cdot e^{-\left(\frac{\tau_1}{t}\right)^{\beta_1}}} + \frac{\varphi_1}{E_0} t'^{-m} (t - t')^n \tag{4}$$

In which two additional parameters τ_1 and β_1 are included in the formulation. The resulting equation is then more capable of adhering to the know decrease of initial displacements observed in creep tests and compliances obtained from design codes. In this way, if presmoothing is required, it is possible to modify a given existent creep law if it is unable to adhere to a particular feature of the data under analysis.

3 Methodology

The methodology followed in this work consisted in using the Python programming language to implement an algorithm to perform a non-linear curve fitting of eq. (1) to a set of compliance functions obtained at different loading ages, in order to obtain the parameters of a Generalized Kelvin Chain model. The retardation times were previously defined and followed the recommendations outlined and discussed in section **2.2**. Then, the non-linear curve fitting provided the values of the modulus E_i of each Kelvin unit. The fitting for each age was performed separately and the space of solution was bounded to $(0, \infty)$ to prevent negative modulus values, which, although not mathematically incorrect, are physically implausible. The curve fitting was performed with the method *curve_fit* from the SciPy library, which performed a non-linear least square fitting using the Trust Region Reflective optimization algorithm, adequate for bounded problems, as is the case implemented in this work [24].

The aging compliance data used to test the Generalized Kelvin Chain fitting algorithm were obtained from an algorithm implemented to generate compliances per Eurocode 2 [6] and *fib* Model Code 2010 [7]. To generate the exemplificatory aging compliances shown in section **4** , the following inputs were used: Eurocode 2 creep model; a cross-sectional area of 62500 mm^2; a notional size of 1000 mm; a concrete class “C50”; a cement type “42.5N”; and an ambient temperature of 20ºC; relative humidity of 80%. The total creep time was 365 days. The loading ages and data points in each compliance curve were sampled uniformly in the logarithmic scale.

A presmoothing feature was also implemented in the code, with the possibility to fit eqs. (3) and (4), and the same framework can be used to implement any other creep law. All the developed algorithms (Generalized Kelvin Chain model fitting, creep law presmoothing, and compliance generation according to Eurocode 2 and *fib* Model Code 2010 design codes) are open-source and available at a GitHub repository [25].

4 Results and discussion

An example of the functionalities of the developed algorithm (*e.g.*, the non-linear curve fitting for obtaining the Generalized Kelvin Chain model and the presmoothing) is summarized in Figure 2. In this example, an artificial random noise was added to aging compliances computed from Eurocode 2 (labelled as "EC2-Xd"). The noise followed a normal distribution with a zero average and a 1% standard deviation dependent on the magnitude of each data point. The artificially noisy aging compliances were then presmoothed with Double Power Law curves (labelled as "DPL-Xd") accordingly to eq. (4), which were then used to fit eq. (1) in order to obtain the full Generalized Kelvin Chain model curves (labelled as "KC-Xd"). A total of eight loading ages were included in the example. From Figure 2, one can observe that the superposition and smoothness of the Double Power Law and Generalized Kelvin Chain curves is guaranteed even though the original compliance curves are wavy due to the added random noise. The excellent superposition between the smoothed and Kelvin Chain compliance curves, which are even difficult to distinguish visually, indicates the developed fitting algorithm has a good performance and suggests it can be used to derive Generalized Kelvin Chain models from compliances computed from design codes or data contaminated with noises or other potential issues. This workflow emulated what could be done when one is working with purely experimental data, which can not only suffer from random errors but also partial data loss, obstacles that can be circumvented with presmoothing. The Generalized Kelvin Chain model used was comprised of nine Kelvin units. This number was automatically chosen after following the recommendations given in section **2.2** about the selection of retardation times.

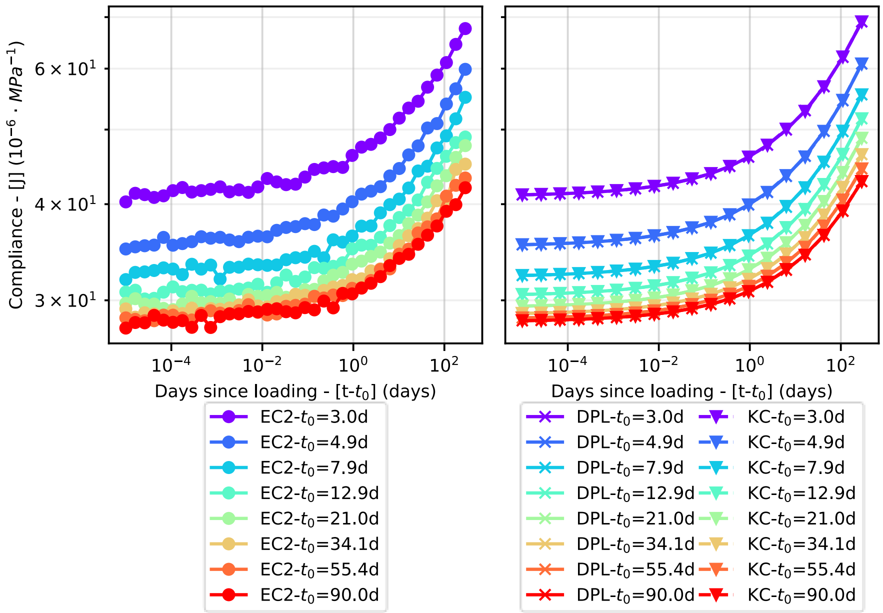


Figure 2 Aging compliances from Eurocode 2 (EC2-Xd) with an added artificial noise, their smoothing with a Double Power Law given by eq. (4) (DPL-Xd), and their representation with Generalized Kelvin Chain models (KC-Xd), in which X represents the age of loading (X=3.0, 4.9, 7.9, …, 90.0 days).

Figure 3 shows the discrete retardation spectrum associated to the Generalized Kelvin Chain model shown in shown in Figure 2. The retardation times presented in the spectrum were automatically defined after following the recommendations outlined in section 2.2, and so the smallest retardation time, τ_1, is equal to 10^{-9} days, and the highest is equal to 300 days, which was the smallest value in a equidistant logarithmic sampling starting at the second smallest retardation time, τ_2, defined as 3×10^{-5} days based on the first time instant defined in the compliance function, to be equal or higher than half of the maximum creep time of interest, which was 365 days. The physical interpretation Figure 3 requires recalling that, in eq. (1), the modulus is a denominator in the series, and so the higher its value, the less contribution the associated Kelvin unit provides to the overall compliance. Figure 3 indicates that higher retardation times, associated to slower creep mechanisms, contribute more to the compliance function that smaller retardation times, associated to quick creep mechanism. Indeed, concrete is known to present a viscoelastic phenomenon that evolves significantly along time, with a retardation spectrum whose higher contributions are within scales of 10^0 to 10^3 days [20]. Also, as aging progresses, it is observed a consistent increase in the modulus of all chains, indicating an overall smaller compliance as expected and observed in Figure 2.

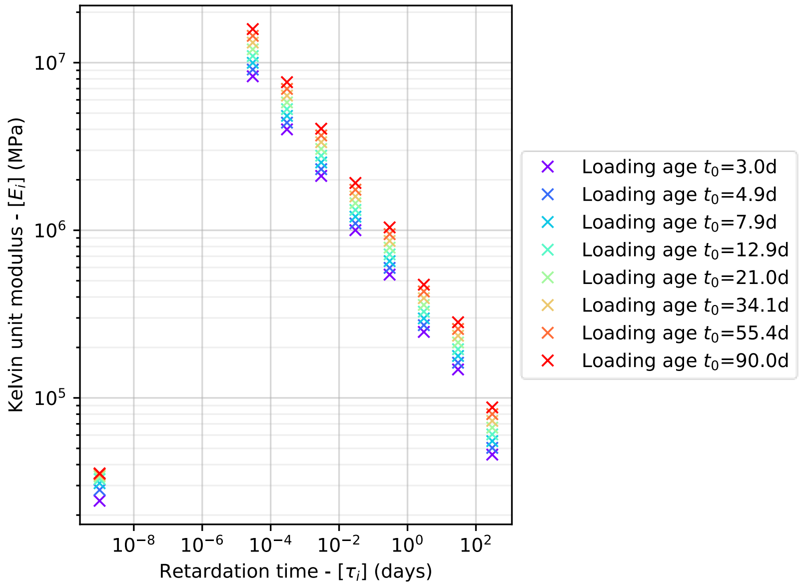


Figure 3 Discrete retardation spectrum associated to the aging Generalized Kelvin Chain model.

5 Conclusion

The algorithm developed in this work was shown to be capable of providing a Generalized Kelvin Chain model from aging compliances curves obtained from models given by Eurocode 2 [6] and *fib* Model Code 2010 [7]. The use of presmoothing was exemplified by deriving a Generalized Kelvin Chain model from a set of compliances in which noise was artificially added. While this strategy of curve fitting is known to be generally ill-posed, if recommendations found on the literature of concrete creep are followed, as the ones outlined in section **2.2**, the fitting procedure was successful. The algorithm is completely open-source and available at [25].

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